Total domination game on ladder graphs

Karnchna Charoensitthichai and Chalermpong Worawannootai

Department of Mathematics, Faculty of Science,
Silpakorn University, Sanam Chandra Palace Campus, Mueang, Nakhon Pathom, 73000 Thailand

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Abstract

The total domination game is played on a simple graph \( G \) by two players, named Dominator and Staller. They alternately select a vertex of \( G \); each chosen vertex totally dominates its neighbors. In this game, each chosen vertex must totally dominate at least one new vertex not totally dominated before. The game ends when all vertices in \( G \) are totally dominated. Dominator's goal is to finish the game as soon as possible, and Staller's goal is to prolong it as much as possible. The game total domination number is the number of chosen vertices when both players play optimally. There are two types of such number, one where Dominator starts the game and another where Staller starts the game. In this paper, we determine the game total domination numbers of the ladders, the circular ladders and the Möbius ladders.

Keywords: total domination game, Game total domination number, ladder, circular ladder, Möbius ladder

1. Introduction

The domination game in graphs was introduced by Brešar, Klavžar, and Rall (2010), where the original idea is attributed to Henning in 2003 and the game has been extensively studied afterwards in (Brešar, Dorbec, Klavžar, & Košmrlj, 2014; Dorbec, Košmrlj, & Renault, 2015; Knittersley, West, & Zamani, 2013; Košmrlj, 2017) and elsewhere. A subset \( S \) of vertices in a graph \( G \) is called a dominating set if every vertex not in \( S \) is adjacent to some vertex in \( S \). If a vertex \( u \in S \) is adjacent to a vertex \( v \in V(G) \), we say that \( u \) dominates \( v \). Domination game is played on a graph \( G \) by two players, Dominator and Staller, who alternate taking turns choosing a vertex from \( G \) such that whenever a vertex is chosen, at least one additional vertex is dominated. Playing a vertex will make all vertices in its close neighborhood dominated. The game ends when the chosen vertices from two players form a dominating set i.e. all vertices are dominated. Dominator's goal is to finish the game as soon as possible, and Staller's goal is to prolong it as much as possible. Note that domination game is a game without winner or loser but the players want to play optimally according to their purposes. The game domination number is the size of the dominating set of chosen vertices when both players play optimally, denoted by \( \gamma(G) \) when Dominator starts the game and denoted by \( \gamma'(G) \) when Staller starts the game.

There are many results about game domination numbers on various families of graphs. Brešar et al. (2010) showed a lower bound on the game domination number of an arbitrary Cartesian product of two graphs. Košmrlj (2017) determined the game domination numbers for paths and cycles. Dorbec et al. (2015) showed how the game domination number of the union of two graphs from a certain family corresponds to the game domination numbers of the initial graphs. Rukkasakchai et al. (2019) showed the game domination numbers of a disjoint union of paths and cycles.

In this paper, we study the total version of domination game which is called the total domination game. A set \( S \) of vertices of a graph \( G \) is a total dominating set, abbreviated TD-set, if every vertex of \( G \) is adjacent to some vertex in \( S \). If a vertex \( u \in S \) is adjacent to another vertex \( v \in V(G) \), we say that \( u \) totally dominates \( v \). Note that in total domination a vertex does not totally dominate itself so it is required that there is no isolated vertex in the graph. The total domination number of a graph \( G \) is the minimum cardinality of a total dominating set of \( G \), denoted by \( \gamma_t(G) \).
For any graph $G$ which has no isolated vertex, $\gamma_t(G)$ exists and $\gamma_t(G) \geq 2$.

The total domination game was introduced by Henning, Klavžar, and Rall (2015). This game is played on a graph $G$ by two players, Dominator and Staller, who alternate taking turns to choose a vertex from a graph $G$ Each chosen vertex must totally dominate at least one new vertex not totally dominated before. The game ends when the set of vertices chosen by the two players is a total dominating set in $G$ Dominator's goal is to finish the game as soon as possible, and Staller's goal is to prolong it as much as possible. The game total domination number is the size of the total dominating set of chosen vertices when both players play optimally, denoted by $\gamma'_t(G)$ when Dominator starts the game and denoted by $\gamma'_s(G)$ when Staller starts the game.

Henning et al. (2015) showed a bound of game total domination numbers of a graph in terms of total domination number. More precisely, if $G$ is a graph on at least two vertices, then $\gamma_t(G) \leq \gamma'_s(G) \leq 2\gamma_t(G) - 1$. They also show that the values of the two types of game total domination number of a graph differ by at most 1.

Total domination game played on various families of graphs have been studied. Dorbec and Henning (2016) determined the game total domination numbers for cycles and paths. Henning and Rall (2017) showed that if $G$ is a forest with no isolated vertex, then $\gamma'_t(G) \leq \gamma'_t(G)$. They also characterized the trees with equal total domination and game total domination number.

In order to prove that certain circular ladders are total domination game critical, Henning and Klavžar (2018) determined the game total domination numbers of the circular ladder of order $4k$ and the game total domination numbers of the Möbius ladder of order $4k$ for $k \geq 1$. In this paper, we complement this last result by determining the game total domination numbers of all ladders, circular ladders and Möbius ladders.

2. Preliminaries

A path $P_n$ is a graph whose vertices can be listed in the order $v_1,v_2,...,v_n$ such that $v_i$ and $v_{i+1}$ are adjacent where $i = 1,2,...,n-1$. A cycle $C_n$ is a graph whose vertices can be listed in the order $v_1,v_2,...,v_n$ such that $v_i$ and $v_{i+1}$ are adjacent, and $v_1$ and $v_{i+1}$ are adjacent where $i = 1,2,...,n-1$. The Cartesian product of graphs $G$ and $H$, denoted by $G \Box H$, is the graph whose vertex set is the Cartesian product $V(G) \times V(H)$, two vertices $(u,u')$ and $(v,v')$ are adjacent in $G \Box H$, if and only if either $u = v$ and $u'$ is adjacent to $v'$ in $H$, or $u' = v'$ and $H$ is adjacent to $V$ in $G$. The disjoint union of $m$ copies of a graph $G$ is denoted by $mG$.

To determine the game total domination numbers of the three families of ladder graphs, we will show that playing the total domination game on each such graph is equivalent to playing the domination game on a certain disjoint union of paths or cycles. Here we recall some useful results of domination game of the related graphs.

**Lemma 1.** (Košmrlj, 2017) The game domination numbers of $P_n$ and $C_n$ are given by

$$
\gamma'_t(P_n) = \gamma'_s(C_n) = \begin{cases} \frac{n}{2} - 1, & n = 3 \pmod{4} \\ \frac{n}{2}, & \text{otherwise} \end{cases},
$$

$$
\gamma'_s(P_n) = \begin{cases} \frac{n}{2} - 1, & n = 2 \pmod{4} \\ \frac{n}{2}, & \text{otherwise} \end{cases},
$$

and

$$
\gamma'_s(C_n) = \begin{cases} \frac{n-1}{2} - 1, & n = 2 \pmod{4} \\ \frac{n-1}{2}, & \text{otherwise} \end{cases}.
$$

**Lemma 2.** (Rukkasakchai et al., 2019) The game domination numbers of $2P_n$ are given by

$$
\gamma'_t(2P_n) = \gamma'_s(2P_n) = \begin{cases} n+1, & n = 1 \pmod{4} \\ n, & \text{otherwise} \end{cases}.
$$

**Lemma 3.** (Rukkasakchai et al., 2019) The game domination numbers of $2C_n$ are given by

$$
\gamma'_t(2C_n) = \begin{cases} n-1, & n = 2,3 \pmod{4} \\ n, & \text{otherwise} \end{cases}
$$

and

$$
\gamma'_s(2C_n) = \begin{cases} n-2, & n = 2 \pmod{4} \\ n-1, & n = 1,3 \pmod{4} \\ n, & n = 0 \pmod{4} \end{cases}.
$$

3. Ladder Graphs

For $n \geq 1$, the ladder $L_n$ is the Cartesian product $P_n \Box P_2$. Throughout the paper, we let $V(L_n) = \{x_1,x_2,...,x_n, y_1,y_2,...,y_n\}$ and $E(L_n) = \{x_iy_j \mid 1 \leq i,j \leq n \text{ and } |i-j| \leq 1\}$. The ladders $L_n$ and $L_3$ are shown in Figure 1(a)(b). In this section, we determine the game total domination numbers of the ladder $L_n$. To do so, we show that playing the total domination game on $L_n$ is equivalent to playing the domination game on $2P_n$.

**Theorem 4.** The game total domination numbers of the ladder $L_n$ are given by

$$
\gamma'_t(L_n) = \gamma'_s(L_n) = \begin{cases} n+1, & n = 1 \pmod{4} \\ n, & \text{otherwise} \end{cases}.
$$
Proof. Let \( G = L_n \). Let \( H = 2P_n \) be the disjoint union of the paths \( u_1u_2 \ldots u_n \) and \( v_1v_2 \ldots v_n \). The graphs \( 2P_4 \) and \( 2P_5 \) are shown in Figure 1(c) and (d). We define a bijection \( f \) from \( V(G) \) to \( V(H) \) by
\[
f(x) = u_i, \quad f(y) = v_i
\]
for \( i \in \{1, 2, \ldots, n\} \). For convenience, we define the reflections on \( V(G) \) and on \( V(H) \) by
\[
x_i = y_i, \quad y_i = x_i, \quad u_i = v_i, \quad v_i = u_i
\]
for \( i \in \{1, 2, \ldots, n\} \). Consider the total domination game on \( G \) and the domination game on \( H \). Recall that in total domination game, playing a vertex \( a \) will totally dominate its open neighborhood \( N(a) \) while in domination game, playing a vertex \( a \) will dominate its closed neighborhood \( N(a) \). Observe that \( f \) and \( f^i \) preserve domination, that is \( f(N_G(a)) = N_H(f(a)) \) and \( f^{-1}(N_H(b)) = N_G(f^{-1}(b)) \). Therefore, if \( a_1, a_2, \ldots, a_n \) is a sequence of moves played in \( G \), then \( f(a_1), f(a_2), \ldots, f(a_n) \) is a sequence of moves played in \( H \), and if \( b_1, b_2, \ldots, b_n \) is a sequence of moves played in \( H \), then \( f^{-1}(b_1), f^{-1}(b_2), \ldots, f^{-1}(b_n) \) is a sequence of moves played in \( G \). In other words, the two games are essentially the same so \( \gamma_G(G) = \gamma_H(H) \) and \( \gamma'_G(G) = \gamma'_H(H) \). The result follows from Lemma 2.

4. Circular Ladder Graphs

For \( n \geq 3 \), the circular ladder \( CL_n \) is the Cartesian product \( C_n \times \mathbb{N}_2 \). Here \( CL_n \) is the graph obtained from the ladder \( L_n \) by adding the edges \( x_iy_i, y_iy_{i+1} \) when \( n \) is odd, and the edges \( x_iy_i, y_ix_{i+1} \) when \( n \) is even. The circular ladders \( CL_4 \) and \( CL_5 \) are shown in Figure 2(a) and (b). In this section, we determine the game total domination numbers of the circular ladders. To do so, we show that (i) playing the total domination game on \( CL_{2k+1} \) is equivalent to playing the domination game on \( C_{4k+2} \); and (ii) playing the total domination game on \( CL_{2k} \) is equivalent to playing the domination game on \( 2C_{2k} \).

![Image of graphs](image_url)

Figure 1. The ladders \( L_n \) and \( L_5 \), and the graphs \( 2P_4 \) and \( 2P_5 \).

![Image of graphs](image_url)

Figure 2. The circular ladders \( CL_4 \) and \( CL_5 \), and the graphs \( 2C_4 \) and \( C_{10} \).

**Theorem 5.** The game total domination numbers of the circular ladder \( CL_n \) are given by
\[
\gamma_G(CL_n) = \begin{cases} n - 1; & n = 2 \text{(mod 4)} \\
\phantom{=} n; & \text{otherwise} \end{cases}
\]
and
\[
\gamma'_G(CL_n) = \begin{cases} n - 2; & n = 2 \text{(mod 4)} \\
\phantom{=} n - 1; & n = 1, 3 \text{(mod 4)} \\
\phantom{=} n; & n = 0 \text{(mod 4)} \end{cases}
\]

**Proof.** Let \( G = CL_n \). Let \( H \) be the disjoint union of the cycles \( u_1u_2 \ldots u_nu_1 \) and \( v_1v_2 \ldots v_nv_1 \) if \( n \) is even; otherwise let \( H \) be the cycle \( u_1u_2 \ldots u_nv_1v_2 \ldots v_nu_1 \). The graphs \( 2C_4 \) and \( C_{10} \) are shown in Figure 2(c) and (d). We define a bijection \( f \) from \( V(G) \) to \( V(H) \) by
\[
f(x) = u_i, \quad f(y) = v_i
\]
for \( i \in \{1, 2, \ldots, n\} \).

In the similar manner to the proof of Theorem 4, we see that the total domination game on \( G \) and the domination game on \( H \) are essentially the same so \( \gamma_G(G) = \gamma_H(H) \) and \( \gamma'_G(G) = \gamma'_H(H) \). The result follows from Lemma 1 and Lemma 3.
5. Möbius Ladder Graphs

For \( n \geq 2 \), the Möbius ladder \( ML_n \) is the graph obtained from the ladder \( L_n \) by adding the edges \( x_nx_1, y_ny_1 \) when \( n \) is even, and the edges \( x_nx_1, y_ny_1 \) when \( n \) is odd. The Möbius ladders \( ML_4 \) and \( ML_5 \) are shown in Figure 3(a)(b). In this section, we determine the game total domination numbers of the Möbius ladder \( ML_n \). To do so, we show that (i) playing the total domination game on \( ML_{2k+1} \) is equivalent to playing the domination game on \( 2C_{2k+1} \); and (ii) playing the total domination game on \( ML_{2k} \) is equivalent to playing the domination game on \( C_{4k} \).

![Möbius Ladder Graphs](image)

**Figure 3.** The Möbius ladders \( ML_4 \) and \( ML_5 \), and the graphs \( C_8 \) and \( 2C_4 \).

**Theorem 6.** The game total domination numbers of the Möbius ladder \( ML_n \) are given by

\[
\gamma_t(ML_n) = \begin{cases} 
  n-1; & n \equiv 3 \pmod{4} \\
  n; & \text{otherwise}
\end{cases}
\]

and

\[
\gamma'_t(ML_n) = \begin{cases} 
  n-1; & n \equiv 1,3 \pmod{4} \\
  n; & \text{otherwise}
\end{cases}
\]

**Proof.** Let \( G = ML_n \). Let \( H \) be the disjoint union of the cycles \( u_1u_2 \ldots u_nu_1 \) and \( v_1v_2 \ldots v_nv_1 \) if \( n \) is odd; otherwise let \( H \) be the cycle \( u_1u_2 \ldots u_nv_1v_2 \ldots v_nu_1 \). We define a bijection \( f \) from \( V(G) \) to \( V(H) \) by

\[
f(x_i) = u_i, \quad f(y_i) = v_i
\]

for \( i \in \{1, 2, \ldots, n\} \).

In the similar manner to the proof of Theorem 4, we see that the total domination game on \( G \) and the domination game on \( H \) are essentially the same so \( \gamma_t(G) = \gamma_t(H) \) and \( \gamma'_t(G) = \gamma'_t(H) \). The result follows from Lemma 1 and Lemma 3.

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**References**


