Formulae of the Frobenius number in relatively prime three Lucas numbers

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Abstract

In this paper, we find the explicit formulae of the Frobenius number for numerical semigroups generated by relatively prime three Lucas numbers \( L_i, L_{i+1}, \) and \( L_{i+2} \) for given integers \( i \geq 3, l \geq 4. \)

Keywords: Frobenius number, Lucas numbers, Fibonacci numbers

1. Introduction

Let \( a_1, a_2, \ldots, a_n (n \geq 2) \) be integers. Any expression of the form \( c_1a_1 + c_2a_2 + \ldots + c_na_n \) where \( c_1, c_2, \ldots, c_n \) are integers, is called a linear combination of \( a_1, a_2, \ldots, a_n. \)

Given positive integers \( a_1, a_2, \ldots, a_n (n \geq 2) \) with gcd \( (a_1, \ldots, a_n) = 1, \) the Frobenius Problem is a problem to determine the largest positive integer that cannot be representable as a nonnegative integer combination of \( a_1, \ldots, a_n. \)

Definition The Frobenius number of \( a_1, a_2, \ldots, a_n, \) denoted by \( g(a_1, a_2, \ldots, a_n). \) is the largest integer \( Z \) such that \( Z \neq c_1a_1 + c_2a_2 + \ldots + c_na_n \) for all nonnegative integers \( c_1, c_2, \ldots, c_n. \)

For example, \( g(3, 5) = 7, \) \( g(6, 9, 20) = 43. \)

The Frobenius Problem is well known as the coin problem that asks for the largest monetary amount that cannot be obtained using only coins in the set of coin denominations which has no common divisor greater than 1. This problem is also referred to as the McNugget number problem introduced by Henri Picciotto. There are several applications of the Frobenius Problem, for example, in obtaining upper bounds for the running time of the Shell-sort algorithm, studying partitions of vector spaces and investigating algebraic geometric codes; see Ramírez Alfonsín (2005).

The origin of this problem for \( n = 2 \) was proposed by Sylvester (1884), and this was solved by Sharp (1884) :

\[
g(a_1, a_2) = (a_1 - 1)(a_2 - 1) - 1 = a_1a_2 - a_1 - a_2.\]
In the 21st century, the Frobenius Problem is still an interesting problem. There are several studies associated with this problem, as follows. Marín et al. (2007) investigated the Frobenius number of Fibonacci numbers \( F_i, F_{i+2}, F_{i+k} \) for integers \( i, k \geq 3 \) where \( F_i \) is the \( n \)th term of the Fibonacci sequence defined by \( F_n = F_{n-1} + F_{n-2} \), \( n \geq 3 \) with \( F_1 = 1 \) and \( F_2 = 1 \). They found that

\[
g(F_i, F_{i+2}, F_{i+k}) = \begin{cases} (F_i - 1)F_{i+2} - F_i(rF_{i+2} + 1), & \text{if } r = 0 \text{ or } r \geq 1 \text{ and } F_{i+2} < (F_i - rF_i)F_{i+2}, \\ (rF_i - 1)F_{i+2} - F_i((r-1)F_{i+2} + 1), & \text{otherwise,} \end{cases}
\]

where \( r = \left\lfloor \frac{F_i - 1}{F_k} \right\rfloor \) for \( r, k \geq 3 \). Later on, Yiğh and Kýper (2008) established the Frobenius number involving Lucas numbers \( L_n \) defined by \( L_n = L_{n-1} + L_{n-2} \), \( n \geq 3 \) with \( L_1 = 1 \) and \( L_2 = 2 \). They found the following formulae:

\[
g(L_i, L_{i+2}, L_{i+k}) = L_iL_{i+k} - L_{i+k} - L_{i+k} - L_i \quad \text{for } i, k \geq 2,
\]
\[
g(L_i, L_{i+k}, L_{i+k}) = L_i \left\lfloor \frac{L_i - 2}{2} \right\rfloor + L_{i+k}(L_i - 1) \quad \text{for } i \geq 3,
\]

and

\[
g(L_i, L_{n} + 2L_{n} + 1\right\rfloor + L_{n} + L_{n} - 1 \quad \text{for } i \geq 1.
\]

Moreover, Ong and Ponomarenko (2008) solved the Frobenius Problem for sets of the form \( \{m^2, m^{k-1}n, m^{k-2}n^2, \ldots, n^k\} \), where \( m, n \) are relatively prime positive integers:

\[
g(m^2, m^{k-1}n, m^{k-2}n^2, \ldots, n^k) = n^{k-1}(m^2 - m - n) + (n - 1)m^{k-1}(m^{k-1} - n^{k-1})
\]

for any positive integer \( k \). Gil et al. (2015) found the Frobenius number of primitive Pythagorean triples:

\[
g(m^2 - n^2, 2mn, m^2 + n^2) = (m-1)(m^2 - n^2) + (m-1)(2mn) - (m^2 + n^2).
\]

Recently, Tripathi (2017) gave an exact formula for \( g(a_1, a_2, a_3) \), where \( a_1, a_2, a_3 \) are pairwise coprime positive integers. His results are divided into several cases and are complicated, so we do not record them here.

In a recent paper, we investigate the Frobenius number \( g(L_i, L_{i+k}, L_{i+k}) \) for integers \( i \geq 3 \), \( k \geq 4 \) by using the idea in Marín et al. (2007) and generalize the work of Yiğh and Kýper (2008). Our work needs the well-known Theorem of Brauer and Shockley (1962) stated as follows:

**Theorem A.** Let \( 1 < a_1 < \ldots < a_n \) be integers such that \( \gcd(a_1, \ldots, a_n) = 1 \).

Let \( B = \{a_1x_1 + \ldots + a_nx_n \mid x_i \in \mathbb{N} \cup \{0\} \text{ for all } i = 1, 2, \ldots, n\} \). Then
\[ g(a_1, \ldots, a_n) = \max_{l \in \{1, 2, \ldots, n\}} \{ t_l \} - a_1, \]

where \( t_l \) is the smallest positive integer congruent to \( l \) modulo \( a_i \) and \( t_l \in B \).

Note that Theorem A can give the value for \( g(a_1, \ldots, a_n) \); however, the formula is not in closed form and it is difficult to find \( t_l \) for each \( l \). In our work, we are able to give an explicit formula for \( g(L_1, L_{i+1}, L_{i+1}) \).

2. Necessary Lemmas

Before investigating the value of \( g(L_1, L_{i+1}, L_{i+1}) \) for \( i \geq 3 \), \( l \geq 4 \), we establish some lemmas. By Theorem A, for fixed integers \( i \geq 3 \), \( l \geq 4 \), we get

\[ g(L_1, L_{i+1}, L_{i+1}) = \max_{k \in \{2, \ldots, i\}} \{ t'_k \} - L_l \]

where \( t'_k \) is the smallest positive integer congruent to \( k \) modulo \( L_1 \) and \( t'_k = xL_{i+1} + yL_{i+1} \) for some \( x, y \geq 0 \). Then we shall construct the Table 1, denoted by \( T_1 \), having entries \( t_{x,y} = xL_{i+1} + yL_{i+1} \) for integers \( x, y \geq 0 \). Since

\[ L_{i+1} = L_{i+1}F_i + L_{i+1}F_{i-2} = L_{i+1}(F_i - F_{i-2}) + (L_{i+1} - L_1)F_{i-2} = F_{i+1}L_{i+1} - F_{i-2}L_1, \]

we get

\[ t_{x,y} = xL_{i+1} + yL_{i+1} = xL_{i+1} + y(F_{i+1}L_{i+1} - F_{i-2}L_1) = (x + yF_i)L_{i+1} - yF_{i-2}L_1. \]

Thus the table \( T_1 \) can be represented as the table \( T_1 \).

Table 1. \( T_1 : t_{x,y} = xL_{i+1} + yL_{i+1} \) for \( x, y \geq 0 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>\ldots</th>
<th>( r )</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>( L_{i+1} )</td>
<td>( 2L_{i+1} )</td>
<td>\ldots</td>
<td>( rL_{i+1} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>( 2L_{i+1} )</td>
<td>( 2L_{i+1} + L_{i+1} )</td>
<td>( 2L_{i+1} + 2L_{i+1} )</td>
<td>\ldots</td>
<td>( 2L_{i+1} + rL_{i+1} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( 3L_{i+1} )</td>
<td>( 3L_{i+1} + L_{i+1} )</td>
<td>( 3L_{i+1} + 2L_{i+1} )</td>
<td>\ldots</td>
<td>( 3L_{i+1} + rL_{i+1} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>\ldots</td>
<td>( \vdots )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( F_i - 1 )</td>
<td>( (F_i - 2)L_{i+1} )</td>
<td>( (F_i - 2)L_{i+1} + L_{i+1} )</td>
<td>( (F_i - 2)L_{i+1} + 2L_{i+1} )</td>
<td>\ldots</td>
<td>( (F_i - 2)L_{i+1} + rL_{i+1} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( F_i )</td>
<td>( F_iL_{i+1} )</td>
<td>( F_iL_{i+1} + L_{i+1} )</td>
<td>( F_iL_{i+1} + 2L_{i+1} )</td>
<td>\ldots</td>
<td>( F_iL_{i+1} + rL_{i+1} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>\ldots</td>
<td>( \vdots )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From now on, we define the set \( T_{F_i-1} \) to contain the first \( F_i - 1 \) entries of all columns in Table 2: \( T_1 \). That is

\[ T_{F_i-1} = \{ t_{x,y} \mid 0 \leq x \leq F_i - 1 \ \text{and} \ y \geq 0 \}. \]
Throughout the paper, we set $r = \left\lfloor \frac{L_1 - 1}{F_i} \right\rfloor$ and $L_1 - 1 = rF_i + q$ for some integer $0 \leq q \leq F_i - 1$. Let $T_{F_i-1,r}$ be the set that contains the first $F_i - 1$ entries of columns $0, 1, 2, \ldots, r - 1$ and the first $q$ entries of column $r$, i.e.,

$$T_{F_i-1,r} = \{ t_{i,j} \mid 0 \leq x \leq F_i - 1 \text{ and } 0 \leq y \leq r - 1 \} \cup \{ t_{0,j}, t_{1,j}, \ldots, t_{q,j} \}.$$

**Lemma 1.**

(i) The set $T_{F_i-1,r}$ is a complete system of residues modulo $L_i$.

(ii) In the table $T_r$, $t_{m,n} \leq t_{j,k}$ for all $m \leq j$ and $n \leq k$. Moreover, $t_{m+1,n} < t_{m,n+1}$ for all $0 \leq m,n \leq F_i - 2$.

**Proof.**

(i) For each $t_{i,j} = (x + yF_i)l_{i+2} - yF_i, L_i \in T_{F_i-1,r}$, we have $0 \leq x + yF_i \leq q + rF_i = L_i - 1$. Since $(L_i, l_{i+2}) = 1$, $T_{F_i-1,r}$ is a complete system of residues modulo $L_i$.

(ii) Recall that $t_{m,n} = ml_{i+2} + nL_{i+1}$ and $t_{j,k} = jL_{i+2} + kL_{i+1}$. It is obvious that for $m \leq j$, $t_{m,n} \leq t_{j,n}$ and for $n \leq k$, $t_{m,n} \leq t_{m,k}$. Therefore, $t_{m,n} \leq t_{j,k}$ for all $m \leq j$ and $n \leq k$. For $0 \leq m,n \leq F_i - 2$, we have $t_{m+1,n} - t_{m,n} = (F_i - 1)L_{i+2} - F_i - 2L_i > 0$.

We define $t_i$ as follows:

- $t_{0,0} = t_{0,1} \quad t_{1,0} = t_{1,1} \quad t_{2,0} = t_{2,1} \quad t_{3,0} = t_{3,1} \quad \ldots \quad t_{r,0} = t_{r,1} \quad \ldots$
- $t_{1,1} = t_{1,2} \quad t_{2,1} = t_{2,2} \quad t_{3,1} = t_{3,2} \quad \ldots \quad t_{r,1} = t_{r,2} \quad \ldots$
- $t_{2,0} = t_{2,1} \quad t_{2,2} = t_{2,3} \quad \ldots \quad t_{2,r} = t_{2,r} \quad \ldots$
- $t_{3,1} = t_{3,2} \quad t_{3,3} = t_{3,4} \quad \ldots \quad t_{3,r} = t_{3,r} \quad \ldots$
- $t_{r-1,0} = t_{r-1,1} \quad t_{r-1,1} = t_{r-1,2} \quad \ldots \quad t_{r-1,r} = t_{r-1,r} \quad \ldots$

The elements of $T_{F_i-1,r}$ can be represented as $t_i = xL_{i+2} - x^2F_iL_i$ for $x = 0, 1, \ldots$.

**Lemma 2.** Let $t_{i,j}$ be an entry of $T_r$ and $t_{i,j} \notin T_{F_i-1,r}$. Then there exist $t_{i,j} \in T_{F_i-1,r}$ such that $t_{i,j} = t_{i,j} \pmod{L_i}$ and $t_{i,j} > t_{i,j}$.
Proof. By the definition of $T_{e}$ given above, the set $T_{e-1,r}$ can be written as

$$T_{e-1,r} = \{ t_{0}, \ldots, t_{F_{r-1}}, t_{F_{r}}, t_{2F_{r}}, \ldots, t_{3F_{r-1}} \}.$$  

We will consider two cases as follows.

Case 1: $t_{u,v} \in T_{F_{r-1},r} \setminus T_{e-1,r}$

Then $t_{u,v} = t_{aL+b}$ for some integer $a \geq 1$ and $0 \leq b \leq L_{e} - 1$. We see that

$$t_{aL+b} = (aL_{e} + b)L_{i+2} = \frac{aL_{e} + b}{F_{i}} F_{i+2} L_{e} = bL_{i+2} - \frac{b}{F_{i}} F_{i+2} L_{e} = t_{b} \pmod{L_{e}}.$$ 

Since $0 \leq b \leq L_{e} - 1$, $t_{b} = t_{aL+b}$ for some $x, y$. That is, $t_{u,v} = t_{x,y} \pmod{L_{e}}$. Next, we will show that $t_{aL+b} > t_{x,y}$, i.e.,

$t_{aL+b} > t_{b}$. Since $t_{aL+b} \geq t_{b}$ for $a \geq 1$, it is enough to show only that $t_{aL+b} > t_{b}$. Recall that $r = \left\lfloor \frac{L_{e} - 1}{F_{i}} \right\rfloor$ and

$L_{e} - 1 = rF_{i} + q$ for some $0 \leq q \leq F_{i} - 1$. We will consider two subcases depending on the value of $r$.

Subcase 1.1: If $r = 0$, then $L_{e} - 1 < F_{i}$, so $L_{e} + b \leq 2F_{i} - 1$. If $0 \leq L_{e} + b \leq F_{i} - 1$, then both $t_{b}$ and $t_{aL+b}$ are in the first column of the table $T_{i}$. By Lemma 1(ii), we obtain $t_{aL+b} > t_{b}$.

Suppose that $F_{i} \leq L_{e} + b < 2F_{i} - 1$, then $t_{b}$ and $t_{aL+b}$ are in the first and second columns of the table $T_{i}$, respectively. If $L_{e} < F_{i}$, then $L_{e} + b < \frac{F_{i}}{2} + \frac{F_{i}}{2} = F_{i}$, a contradiction. Hence we have $F_{i+2} \leq \frac{F_{i}}{2} \leq L_{e}$. Finally, we have

$$t_{aL+b} - t_{b} = L_{i+2} - F_{i+2} = L_{i+2} \left( L_{i+2} - F_{i+2} \right) > 0,$$

Subcase 1.2: Suppose that $r \geq 1$. Consider

$$t_{aL+b} - t_{b} = L_{i+2} \left( \left\lfloor \frac{L_{e} + b}{F_{i}} \right\rfloor - \frac{b}{F_{i}} \right) = L_{i+2} \left( \frac{L_{e} - b + 1}{F_{i}} \right) - m = \frac{rF_{i} + q + mF_{i} + n + 1}{F_{i}} - m \leq r + 1.$$ 

Write $b = mF_{i} + n$ where $0 \leq n \leq F_{i} - 1$. Since $L_{e} - 1 = rF_{i} + q$ with $0 \leq q \leq F_{i} - 1$, it follows that

$$\left\lfloor \frac{L_{e} + b}{F_{i}} \right\rfloor - \frac{b}{F_{i}} = \frac{L_{e} - b + 1}{F_{i}} - m = \frac{rF_{i} + q + mF_{i} + n + 1}{F_{i}} - m \leq r + 1.$$ 

It is enough to show that $L_{i+2} > (r+1)F_{i+2}$. To this end, we see that

$$L_{i+2} - (r+1)F_{i+2} = L_{i+2} - (r+1)F_{i+2}
= rF_{i} + q + 1 + L_{i+1} - (r+1)F_{i+2}
= r(F_{i} - F_{i+2}) + q + 1 + L_{i+1}
= rF_{i+1} - F_{i+2} + q + 1 + L_{i+1} > 0$$

since $r \geq 1$. 

Case 2: \( t_{x,y} \not\in T_{F_{i,j}} \)

Since \( T_{F_{i,j}} \) is a complete system of residues modulo \( L_i \), there exists \( t_{x,y} \in T_{F_{i,j}} \) such that \( t_{x,y} = t_{x,y} \pmod{L_i} \). Then \( 0 \leq x \leq F_{i,j} - 1 < u \). If \( v \geq y \), by Lemma 1(ii), \( t_{x,y} \leq t_{x,y} < t_{x,y} \). Suppose \( v < y \). Then \( t_{x,y} = t_{x,y} \pmod{L_i} \) implies \( u + vF_i = x + yF_i \pmod{L_i} \). From Lemma 1(i), \( 0 \leq x + yF_i \leq L_i - 1 \), and thus \( u + vF_i = m(x + yF_i) \) for some integer \( m \geq 1 \). Hence \( u + vF_i \geq x + yF_i \). Since \( nF_{i,j} L_i > -yF_{i,j} L_i \), we have \( t_{x,y} > t_{x,y} \).

3. Main Theorem

Theorem. Let \( i \geq 3, l \geq 4 \) be integers and \( r = \left\lfloor \frac{L_i - 1}{F_i} \right\rfloor \). Then

\[
g(L_i, L_{i+2}, L_{i+4}) = \begin{cases} 
(L_i - 1)L_{i+2} - (1 + rF_{i-3})L_i, & \text{if } 1 \leq r, \\
(rF_i - 1)L_{i+2} - (1 + (r - 1)F_{i-3})L_i, & \text{otherwise}.
\end{cases}
\]

Proof. From Theorem A, now we have to consider \( t'_k \) for \( k = 1, 2, \ldots, L_i - 1 \) when \( t'_k \) is the smallest positive integer congruent to \( k \) modulo \( L_i \) and \( t'_k \) can be written as \( xL_{i+2} + yL_{i+4} \) for some integers \( x, y \geq 0 \). Since \( t'_k = xL_{i+2} - \left\lceil \frac{x}{F_i} \right\rceil F_{i+2}L_i \) for \( x = 0, 1, \ldots \).

If \( r = 0 \), by Lemma 2, we have that \( t'_{x} \) is the smallest positive integer congruent to \( k \) modulo \( L_i \) for some integer \( 0 \leq k \leq L_i - 1 \). And we see that \( t'_{x} \) can be represented as a linear combination of \( L_{i+2} \) and \( L_{i+4} \). Hence \( T_{F_{i,j}} = \{ t'_{x} \mid k = 1, 2, \ldots, L_i - 1 \} \). If \( r \geq 1 \), by Lemma 1(ii), then

\[
t_{F_{i,j}} = \max_{0 \leq k < r} \{ t'_{x} \mid t'_{x} \in T_{F_{i,j}} \} \quad \text{for each } i = 0, 1, 2, \ldots, r - 1,
\]

\[
t_{F_{i,j-1}} = \max_{0 \leq k < r - 1} \{ t'_{x} \mid t'_{x} \in T_{F_{i,j}} \},
\]

and

\[
t_{F_{i,j-1}} = \max_{0 \leq k \leq r - 1} \{ t'_{x} \mid t'_{x} \in T_{F_{i,j}} \}.
\]

We will find the necessary condition for \( t_{F_{i,j}} > t_{F_{i,j-1}} \). It is true if and only if \((L_i - 1)L_{i+2} - F_{i+2}L_i > (rF_i - 1)L_{i+2} - (r - 1)F_{i+2}L_i \) that is \((L_i - rF_i)L_{i+2} > F_{i+2}L_i \). Hence we can conclude the result of this theorem.

Example 1. Let \( i = 3 \) and \( l = 5 \). Then \( r = \left\lfloor \frac{L_i - 1}{F_i} \right\rfloor = 0 \). and by our main theorem, we have

\[
g(L_3, L_5, L_7) = g(4, 11, 47) = (L_3 - 1)L_5 - (1 + (0)F_3)L_5 = 3(11) - 1(4) = 29.
\]

We would like to confirm the value of \( g(4, 11, 47) \) by the well-known Theorem A. Since \( g(L_3, L_5, L_7) = g(4, 11, 47) = \max_{k} \{ t'_k \} - 4 \). Then we have to find \( t'_k \) for each \( k = 1, 2, 3 \), that \( t'_k \) is the smallest positive integer congruent to \( k \) modulo

\[
\begin{align*}
g(4, 11, 47) &= 29 \\
\max_{k} \{ t'_k \} &= 29 - 4 = 25.
\end{align*}
\]
$L_k = 4$ and $t^*_i \in B$. We get $t^*_1 = 33, t^*_2 = 22$ and $t^*_3 = 11$. Thus $g(L_3, L_5, L_6) = \max\{33, 22, 11\} - 4 = 29$ which is the same value obtained by our result.

**Example 2.** Take $i = 4$ and $l = 4$. Then $r = \left\lfloor \frac{L_i - 1}{F_i} \right\rfloor = 2$.

and $(L_4 - 2F_4)L_4 > F_2L_4$. Thus

$$g(L_4, L_6, L_8) = g(7,18,47) = (L_4 - 1)L_6 - (1 + 2F_4)L_4 = 87.$$ 

On the other hand, by using Theorem A,

$$g(L_4, L_6, L_8) = g(7,18,47) = \max_{\lambda \in \{1,2,3,4,5,6\}} \{ t^*_i \} - 7.$$ 

We get $t^*_1 = 36, t^*_2 = 65, t^*_3 = 94, t^*_4 = 18, t^*_5 = 47$ and $t^*_6 = 83$. Thus $g(L_4, L_6, L_8) = \max\{ 36, 65, 94, 18, 47, 83 \} - 7 = 87$ which is the same value as above.

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**References**


