Boundedness and continuity of superposition operator on $E_r(p)$ and $F_r(p)$

Assa-aree Sama-ae

Abstract

Sama-ae, A.

Boundedness and continuity of superposition operator on $E_r(p)$ and $F_r(p)


Let $X \in \{E_r(p), F_r(p)\}$, in this research, necessary and sufficient conditions are given for superposition operator to act from $X$ into the space $\ell_1$. Moreover, necessary and sufficient conditions are obtained for superposition operator acting from $X$ into $\ell_1$ to be locally bounded, bounded, and continuous.

Suppose that $P_f$ is a superposition operator which acts from $X$ into $\ell_1$, it is found that

1. $P_f$ is locally bounded if and only if $f$ satisfies the condition $A(2)$,
2. if $P_f$ is bounded then $f$ satisfies the condition $A(2)$,
3. $P_f$ is continuous if and only if $f$ satisfies the condition $A(2)$.

Key words : sequence space, superposition operator, locally bounded function, bounded function, continuous function

M.S. (Mathematics), Department of Mathematics and Computer Science, Faculty of Science and Technology, Prince of Songkla University, Muang, Pattani 94000 Thailand.

Corresponding e-mail : sassaare@bunga.pn.psu.ac.th

Received, 6 December 2001 Accepted, 30 April 2002
The superposition operator plays an important role in studying the theory of representation of orthogonality in additive functionals on sequence spaces. There are many mathematicians trying to characterize the superposition operator acting from a sequence space into $l_1$. For example, Chew (1990) gave necessary and sufficient conditions for superposition operator acting from $w_0$ into $l_1$, and Pluciennik (1990, 1991) characterized local boundedness, boundedness of superposition operator acting from $w_0$ into $l_1$, and continuity of superposition operator acting from $w_0$ and $W_0$ into $l_1$.

In this research, we shall give necessary and sufficient conditions for superposition operator acting from $E_r(p)$ and $F_r(p)$ into $l_1$. Moreover, we shall characterize local boundedness, boundedness, and continuity of superposition operator acting from $E_r(p)$ and $F_r(p)$ into $l_1$.

Let $N$ and $R$ stand for the set of natural numbers and the set of real numbers respectively. Let $x$ be a sequence. For $k \in N$, the $k^{th}$ term of the sequence $x$ is denoted by $x_k$ and we write $x = (x_k) = (x_1, x_2, ... , x_n, ...)$.

Let $\Phi$ denote the space of finite sequences, $S$ be the space of real sequences and

$$ l_1 = \left\{ x = (x_k) \mid \sum_{k=1}^{\infty} |x_k| < \infty \right\} $$

with the norm $\| x \| \equiv \sum_{k=1}^{\infty} |x_k|$ defined by.

$$ \sum_{k=1}^{\infty} |x_k|.$$
Let $|| ||_1 : E_r(p) \rightarrow \mathbb{R}$ be defined by $||x||_1 = \sup |x_k|^{p_k}$. $|| ||_1$ can not be a norm; however, by its properties we can define a metric $d$ on $E_r(p)$ by letting $d(x, y) = ||x-y||_1$ for each $x, y \in E_r(p)$.

Let $|| ||_2 : F_r(p) \rightarrow \mathbb{R}$ be defined by $||x||_2 = \left( \sum_{k=1}^{\infty} |x_k|^{p_k} \right)^{\frac{1}{M_k^{r_k}}}$. We find that $|| ||_2$ is not a norm.

Suppose that $d : F_r(p) \times F_r(p) \rightarrow \mathbb{R}$ is defined by $d(x, y) = ||x-y||_2$ for each $x, y \in F_r(p)$. By using properties of $|| ||_2$ and the inequality,

$$\left( \sum |a_k|^{p_k} + b_k^{p_k} \right)^{\frac{1}{M_k^{r_k}}} \leq \left( \sum |a_k|^{p_k} \right)^{\frac{1}{M_k^{r_k}}} + \left( \sum |b_k|^{p_k} \right)^{\frac{1}{M_k^{r_k}}}$$

shown by Maddox(1970), it is not difficult to verify that $d$ is a metric on $F_r(p)$.

Let $f : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R}$ and $X, Y$ be two sequence spaces. A superposition operator $P_f$ on $X$ is a mapping from $X$ into $S$ defined by $P_f(x) = (f(k, x_k))_{k=1}^{\infty}$. $P_f$ acts from $X$ into $Y$, denoted by $P_f : X \rightarrow Y$, if $P_f(x) \in Y$ for all $x \in X$. We say that the function $f$ satisfies the following conditions:

A(2) if $f(k, \cdot)$ is continuous on $\mathbb{N}$ and $\mathbb{R}$.

It is obvious that $f$ satisfies the condition A(2) if it satisfies the condition A(2).

Let $X = (X, d)$ and $Y = (Y, d^*)$ be two metric sequence spaces. An operator $F : X \rightarrow Y$ is bounded if $F(A)$ is bounded for every bounded subset $A$ of $X$. An operator $F$ is said to be locally bounded at $x_0 \in X$ if there exist $\alpha, \beta > 0$ such that $F(x) \in B_d^*(F(x_0), \beta)$ whenever $x \in B_d(x_0, \alpha)$ and $F$ is locally bounded if $F$ is locally bounded at each $x \in X$.

Throughout this research, we defined $|| ||_1 : E_r(p) \rightarrow \mathbb{R}$ and $|| ||_2 : F_r(p) \rightarrow \mathbb{R}$ by $||x||_1 = \sup |x_k|^{p_k}$ and $||x||_2 = \left( \sum_{k=1}^{\infty} |x_k|^{p_k} \right)^{\frac{1}{M_k^{r_k}}}$ respectively. Suppose that $P_f : X \rightarrow \ell_1$, where $X \in \{E_r(p), F_r(p)\}$, by utilizing the metric induced by previous associated function $|| ||_i$, $i = 1, 2$, we obtain that for each $i$, $P_f$ is bounded if and only if for each $\alpha > 0$ there exists $\beta > 0$ such that $||P_f(x)|| \leq \beta$ whenever $||x||_i \leq \alpha$ and $P_f$ is locally bounded at $x_0 \in X$ if there exists $\alpha, \beta > 0$ such that $||P_f(x)-P_f(x_0)|| \leq \beta$ whenever $||x-x_0||_i \leq \alpha$.

It is easy to verify that if $P_f$ is bounded then it is locally bounded. Finally, we note that
if \( f : R \rightarrow R \) is locally bounded then \( f \) satisfies the condition \( A(2') \). This was justified by Tainchai (1996).

### Superposition operator on \( E_r(p) \)

Firstly, we now give necessary and sufficient conditions for superposition operator acting from \( E_r(p) \) into \( \ell_1 \).

**Theorem 1**

Let \( f : N \times R \rightarrow R \) satisfy the condition \( A(2') \). The superposition operator \( P_f \) acts from \( E_r(p) \) to \( \ell_1 \) if and only if for all \( \alpha > 0 \) there exists a sequence \( (c_k) \in \ell_1 \) such that for each \( k \in N \),

\[
|f(k, t)| \leq c_k \quad \text{whenever} \quad |t|^\frac{r}{r} \leq k^\alpha .
\]

**Proof.**

\((\Leftarrow)\) Let \( x = (x_k) \in E_r(p) \). We now prove that \( P_f \) acts from \( E_r(p) \) to \( \ell_1 \). Since \( (x_k) \in E_r(p) \), there exists \( A > 0 \) such that \( |x_k|^r \leq Ak \) for all \( k \in N \), and thus we get \( |x_k|^\frac{r}{r} = |x_k|^r \left( |x_k|^r \right)^{-\frac{r}{r}} \leq A^\frac{r}{r} k^\frac{r}{r} \leq A^\frac{r}{r} k^\alpha \). It follows from the assumption that there exists a sequence \( (c_k) \in \ell_1 \) such that \( |f(k, x_k)| \leq c_k \) for all \( k \in N \) and consequently we have \( \sum |f(k, x_k)| \leq \sum c_k < \infty \). This implies \( P_f(x) = \epsilon \ell_1 \) and thereby shows that \( P_f \) acts from \( E_r(p) \) to \( \ell_1 \).

\((\Rightarrow)\) Suppose that \( P_f \) acts from \( E_r(p) \) to \( \ell_1 \).

For each \( \alpha > 0 \) and for each \( k \in N \), we define

\[
A(k, \alpha) = \{ t \in R : |t|^\frac{r}{r} \leq (k')^\alpha \}
\]

and \( B(k, \alpha) = \sup \{ |f(k, t)| : t \in A(k, \alpha) \} \).

We note that \( |f(k, t)| \leq B(k, \alpha) \) whenever \( |t|^\frac{r}{r} \leq k' \alpha \). Next, we are going to show that \( B(k, \alpha) \leq c_k \) for each \( \alpha > 0 \). Suppose that there exists \( \alpha_1 > 0 \) such that \( \sum B(k, \alpha_1) = \infty \). Then there is a sequence of positive integers \( n_0 = 0 < n_1 < n_2 < \ldots < n_i < \ldots \) and the least positive integer \( n_i \) such that

\[
\sum_{k = n_{i-1} + 1}^{n_i} B(k, \alpha_1) > 1 .
\]

For each \( i \in N \), there is an \( \epsilon_i > 0 \) such that

\[
\sum_{k = n_{i-1} + 1}^{n_i} B(k, \alpha_1) - \epsilon_i(n_i - n_{i-1}) > 1 .
\]

Let \( i \in N \) be fixed. As \( f \) satisfies the condition \( A(2'), 0 \leq B(k, \alpha_1) < \infty \) for all \( k \in N \) with \( n_{i-1} + 1 \leq k \leq n_i \). It follows from the definition of \( B(k, \alpha_1) \) that for each \( k \in N \) with \( n_{i-1} + 1 \leq k < n_i \) there is \( x_k \in A(k, \alpha_1) \) such that

\[
|f(k, x_k)| > B(k, \alpha_1) - \epsilon_i .
\]

From (1) and (2), for each \( i \in N \),

\[
\sum_{k = n_{i-1} + 1}^{n_i} |f(k, x_k)| > \sum_{k = n_{i-1} + 1}^{n_i} B(k, \alpha_1) - \sum_{k = n_{i-1} + 1}^{n_i} \epsilon_i \\
= \sum_{k = n_{i-1} + 1}^{n_i} B(k, \alpha_1) - \sum_{k = n_{i-1} + 1}^{n_i} \epsilon_i(n_i - n_{i-1}) > 1 .
\]

Thus \( \sum_{k = n_{i-1} + 1}^{n_i} |f(k, x_k)| = \sum_{k = n_{i-1} + 1}^{n_i} |f(k, x_k)| = \infty \), and hence

\[
(f(k, x_k))_{k=1}^\infty \notin \ell_1 .
\]

Since \( x_k \in A(k, \alpha_1) \) for all \( k \in N \) with \( n_{i-1} + 1 \leq k \leq n_i \) and for every \( i \in N \), we obtain that the sequence \( (x_k) \) is in \( E_r(p) \) and therefore

\[
P_f(x) = (f(k, x_k))_{k=1}^\infty \notin \ell_1 ,
\]

which is a contradiction. Accordingly, \( B(k, \alpha) \leq c_k \) for every \( \alpha > 0 \). Put \( c_k = B(k, \alpha) \) for all \( k \in N \). Then for each \( \alpha > 0 \) there exists a sequence \( (c_k) \in \ell_1 \) such that for each \( k \in N \), \( |f(k, t)| \leq c_k \) whenever \( |t|^\frac{r}{r} \leq k' \alpha \).
Theorem 2

Let \( f : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R} \) Suppose that \( P_f \) is a superposition operator which acts from \( E_r(p) \) to \( \ell_1 \). Then \( P_f \) is locally bounded if and only if \( f \) satisfies the condition \( A(2/p) \).

Proof.

\((\Leftarrow)\) Suppose that \( f \) satisfies the condition \( A(2/p) \) and \( P_f \) is superposition operator from \( E_r(p) \) into \( \ell_1 \). Let \( x = (x_k) \in E_r(p) \) be given. We now show that \( P_f \) is locally bounded at \( x \). Let \( \alpha > 0 \) and \( y \in E_r(p) \) with \( \|y - x\| < \alpha \). Then \( \|y\| \leq \alpha + \|x\| \) and this implies \( \|y_k^{1/p}\| \leq k' (\alpha + \|x\|) \) for all \( k \in \mathbb{N} \) and by applying Theorem 1, there is a sequence \((c_k) \in \ell_1\) such that for each \( k \in \mathbb{N} \), \( |f(k, y_k)| \leq c_k \) and hence \( \|P_f(x)\| = \sum_{k=1}^{\infty} |f(k, x_k)| \leq \sum_{k=1}^{\infty} c_k = \|c_k\| \). Therefore the operator \( P_f \) is locally bounded at \( x \), as \( \|P_f(y) - P_f(x)\| < \|P_f(x)\| + \|c_k\| \).

\((\Rightarrow)\) Suppose that \( P_f : E_r(p) \rightarrow \ell_1 \) is locally bounded. We shall prove that \( f \) satisfies the condition \( A(2/p) \); it suffices to show that \( f(k,.) \) is locally bounded for all \( k \in \mathbb{N} \). Let \( k \in \mathbb{N} \) and \( b \in \mathbb{R} \). We define a sequence \( y = (y_k) \) with \( y_k = \begin{cases} b & \text{if } k = n, \\ 0 & \text{if } k \neq n \end{cases} \), and observe that \( y \in E_r(p) \). By the assumption, there exist \( \alpha, \beta > 0 \) such that

\[ \|P_f(x) - P_f(y)\| \leq \beta \text{ whenever } \|x - y\| \leq \alpha. \quad (3) \]

Let \( a \in \mathbb{R} \) with \( |a - b| \leq (k') \alpha \) and let \( x = (x_n) \) be a sequence with \( x_n = \begin{cases} a & \text{if } n = k, \\ 0 & \text{if } n \neq k \end{cases} \). We see that \( x \in E_r(p) \) and \( \|x - y\| = \sup_{n \in \mathbb{N}} \frac{|x_n - y_n|}{n^{1/p}} = \frac{|a - b|}{k'} \leq \alpha \).

It follows from (3) that \( \|P_f(x) - P_f(y)\| \leq \beta \) and thus \( \|f(k, a) - f(k, b)\| \leq \sum_{k=1}^{\infty} |f(n, x_n) - f(n, y_n)| = \|P_f(x) - P_f(y)\| \leq \beta \). Therefore \( f(k,.) \) is locally bounded at \( b \in \mathbb{R} \).

Corollary 3

Let \( f : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R} \). Suppose that \( P_f \) is a superposition operator which acts from \( E_r(p) \) to \( \ell_1 \). Then \( P_f \) is bounded if and only if for each \( \alpha > 0 \) there exist a sequence \((c_k) \in \ell_1\) such that for each \( k \in \mathbb{N} \), \( |f(k, x_k)| \leq c_k \) whenever \( |x_k|^{1/p} \leq k' \alpha \).

Proof.

\((\Rightarrow)\) The result follows directly from Theorem 1 and 2.

\((\Leftarrow)\) Suppose the sufficient condition holds. Let \( \alpha > 0 \) and \( x \in E_r(p) \) with \( \|x\| \leq \alpha \). Thus \( \sup_{k=1}^{\infty} \frac{|x_k|}{k'} \leq \alpha \), that is \( |x_k|^{1/p} \leq k' \alpha \) for every \( k \in \mathbb{N} \). By the assumption there exists a sequence \((c_k) \in \ell_1\) such that for each \( k \in \mathbb{N} \), \( |f(k, x_k)| \leq c_k \). This implies \( \|P_f(x)\| = \sum_{k=1}^{\infty} |f(k, x_k)| \leq \sum_{k=1}^{\infty} c_k = \|c_k\| < \infty \). Accordingly \( P_f \) is bounded.

The following two corollaries are easily verified by utilizing Theorems 1 and 2 and Corollary 3.

Corollary 4

Let \( f : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R} \). Suppose that \( P_f \) is a superposition operator which acts from \( E_r(p) \) to \( \ell_1 \). Then \( P_f \) is bounded if and only if \( f \) satisfies
the condition \( A(2) \).

**Corollary 5**

Let \( f : \mathbb{N} \times \mathbb{R} \to \mathbb{R} \). Suppose that \( P_i \) is a superposition operator which acts from \( E_i(p) \) to \( \ell_i \). Then \( P_i \) is locally bounded if and only if for each \( \alpha > 0 \) there exists a sequence \( (c_k) \in \ell_1 \) such that for each \( k \in \mathbb{N} \), \( |f(k, t)| \leq c_k \) whenever \( |t| \leq k^\alpha \).

**Theorem 6**

Let \( f : \mathbb{N} \times \mathbb{R} \to \mathbb{R} \). The superposition operator \( P_i : E_i(p) \to \ell_i \) is continuous if and only if \( f \) satisfies the condition \( A(2) \).

**Proof**.

(\( \Rightarrow \)) Suppose that \( P_i \) is continuous on \( E_i(p) \). Let \( k \in \mathbb{N} \), \( t_0 \in \mathbb{R} \) and \( \varepsilon > 0 \) be given. As \( P_i \) is continuous at \( t_0 \), there is a \( \delta > 0 \) such that for each \( z \in \mathbb{R} \), \( ||P_i(z) - P_i(t_0)\| < \varepsilon \) whenever \( ||z - t_0\| < \delta \). (4)

Let \( t \in \mathbb{R} \) be such that \( |t - t_0| < (\delta k)^\frac{1}{\alpha} \) and \( y_n = \begin{cases} 0, & n \neq k \\ t, & n = k \end{cases} \). Observe that \( y = (y_n) \in E_i(p) \) and \( \|y - t_0\| = \frac{|t - t_0|}{k^\alpha} < \delta \). Employing (4), we have

\[ \|P_i(y) - P_i(t_0)\| < \varepsilon \] and hence \( |f(k, t) - f(k, t_0)| \leq \varepsilon \). Therefore \( f \) satisfies the condition \( A(2) \).

(\( \Leftarrow \)) Suppose that \( f \) satisfies the condition \( A(2) \). We are going to justify that \( P_i \) is continuous on \( E_i(p) \). Let \( x = (x_n) \in E_i(p) \) and \( \varepsilon > 0 \). It follows from Theorem 1 that for each \( \alpha > 0 \) there exists a sequence \( (c_k) \in \ell_1 \), such that for every \( k \in \mathbb{N} \),

\[ |f(k, t)| \leq c_k \text{ whenever } |t| \leq k^\alpha. \] (5)

As \( (x_k) \) is in \( E_i(p) \), there is \( \beta > 0 \) such that \( |x_k| \leq \left( \frac{k^\beta}{2} \right)^\frac{1}{\alpha} \) for all \( k \in \mathbb{N} \). By (5), there is a sequence \( (c'_k) \in \ell_1 \), such that for all \( k \in \mathbb{N} \),

\[ |f(k, x_k)| \leq c'_k \] (6)

and a sequence \( (c'_k) \in \ell_1 \), such that for each \( k \in \mathbb{N} \),

\[ |f(k, t)| \leq c'_k \text{ whenever } |t| \leq k^\beta. \] (7)

Because \( (c_k) \) and \( (c'_k) \) are in \( \ell_1 \), there is \( N_i \in \mathbb{N} \) such that

\[ \sum_{k=0}^{\infty} c_k < \frac{\varepsilon}{3} \text{ and } \sum_{k=0}^{\infty} c'_k < \frac{\varepsilon}{3}. \] (8)

As a result \( f(k,.) \) is continuous at \( x_k \) for all \( k \in \{1, 2, \ldots, N_i - 1\} \). This implies that there exists \( \delta_k > 0 \) with \( \delta_k < \min \left\{ \delta, \frac{1}{\alpha} \right\} \) such that for each \( k \in \{1, 2, \ldots, N_i - 1\} \) and for every \( t \in \mathbb{R} \),

\[ |f(k, t)| < \frac{\varepsilon}{3(N_i - 1)} \text{ whenever } |t - x_k| < \delta_k. \] (9)

Let \( y = (y_n) \in E(p) \) be such that \( |y - x_k| < \delta \) with \( \delta \leq \min \left\{ \frac{\delta}{k^\alpha} |k = 1, 2, \ldots, N_i - 1\right\} \). Since \( |ly - xl| < \delta \),

sup \( \frac{|y_k - x_k|}{k^\alpha} < \delta \) and hence \( |y_k - x_k| \leq k^\beta \) for all \( k \in \mathbb{N} \). Thus for all \( k \) is in \( \{1, 2, \ldots, N_i - 1\} \),

\[ |y_k - x_k| < (k^\beta)^\frac{1}{\alpha} \leq (\delta_k^\frac{1}{\alpha})^\frac{1}{\alpha} < \delta_k. \] By (9), we have
\(|f(k, y_k) - f(k, x_k)| < \frac{\varepsilon}{3(N_1 - 1)}\) for every \(k \in \{1, 2, \ldots, N_1 - 1\}\), and consequently

\[\sum_{k=1}^{N_1-1} |f(k, y_k) - f(k, x_k)| < \frac{\varepsilon}{3}. \tag{10}\]

As \(|y_k| - |x_k| + |x_k| < \left(\frac{\beta}{2}\right)M + \left(\frac{\beta}{2}\right)M = (k \cdot \beta)^{\frac{M}{N}}\), it follows from (7) that for all \(k \in N\),

\[|f(k, y_k)| \leq c_k. \tag{11}\]

Utilizing (6), (8), and (11) we get

\[\sum_{k=1}^{N_1} |f(k, x_k)| \leq \sum_{k=1}^{N_1} c_k < \frac{\varepsilon}{3} \text{ and } \sum_{k=1}^{N_1} |f(k, y_k)| \leq \sum_{k=1}^{N_1} c_k' < \frac{\varepsilon}{3}. \tag{12}\]

Finally, by using (10) and (12) we obtain that

\[\|P_{f}(y) - P_{f}(x)\| = \sum_{k=1}^{N_1} |f(k, y_k) - f(k, x_k)| \leq \sum_{k=1}^{N_1} |f(k, y_k) - f(k, x_k)| + \sum_{k=1}^{N_1} |f(k, y_k)| + \sum_{k=1}^{N_1} |f(k, x_k)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\]

The proof of Theorem 6 is then complete.

**Theorem 7**

Let \(f : N \times R \rightarrow R\). The superposition operator \(P_{f} : E_{r}(p) \rightarrow \ell\), is uniformly continuous on every bounded subset of \(E_{r}(p)\) if and only if \(f\) satisfies the condition A(2).

**Proof.**

\((\Rightarrow)\) The result is an immediate consequence of Theorem 6.

\((\Leftarrow)\) Assume that \(f\) satisfies A(2). To show that \(P_{f}\) is uniformly continuous on every bounded subset of \(E_{r}(p)\), it is enough to prove that \(P_{f}\) is uniformly continuous on \(B_{\varepsilon}(0, r)\) for all \(r > 0\).

Let \(\sigma > 0\) and \(\varepsilon > 0\) be given. Since \(f\) satisfies the condition A(2), \(f\) satisfies the condition A(2) and hence it follows from Theorem 1 that there exists a sequence \((c_k) \in \ell\), such that for each \(k \in N\),

\[|f(k, t)| \leq c_k \text{ whenever } |t| \leq k \cdot \sigma. \tag{13}\]

Because \((c_k) \in \ell\), there is \(N_1 \in N - \{1\}\) such that \(\sum_{k=1}^{N_1} c_k < \frac{\varepsilon}{3}\). As \(f(k,.)\) is uniformly continuous on \([- (k \cdot \sigma)^{\frac{M}{N}}, (k \cdot \sigma)^{\frac{M}{N}}]\) for every \(k \in \{1, 2, \ldots, N_1 - 1\}\),
there is $0 < \delta_k < 1$ such that for each $k \in \{1, 2, ..., N_1-1\}$ and for all $a, b \in \left( -\frac{(k'\sigma)^M_{\kappa}}{M}, \frac{(k'\sigma)^M_{\kappa}}{M} \right)$,  
$$|f(k, a) - f(k, b)| < \frac{\varepsilon}{3(N_1 - 1)}$$
whenever $|a - b| < \delta_k$. (14)

Let $x$ and $y \in \mathbb{B}_d (0, \sigma)$ be such that $||y - x|| < \delta$, where $\delta < \min \delta_k k = 1, 2, ..., N_1 - 1$. Hence $|x_k| < (k'\sigma)^M_{\kappa}$ and $|y_k| < (k'\sigma)^M_{\kappa}$ for all $k \in \mathbb{N}$ and $|x_k - y_k| < \frac{(k'\sigma)^M_{\kappa}}{k} \leq \delta_k$ for all $k \in \{1, 2, ..., N_1 - 1\}$. Employing (14), we obtain that for all $k \in \{1, 2, ..., N_1 - 1\}$, $|f(k, y_k) - f(k, x_k)| < \frac{\varepsilon}{3}$ and therefore

$$\sum_{k=1}^{N_1-1} |f(k, y_k) - f(k, x_k)| < \frac{\varepsilon}{3}. \quad (15)$$

Because $|x_k| < (k'\sigma)^M_{\kappa}$ and $|y_k| < (k'\sigma)^M_{\kappa}$ for all $k \in \mathbb{N}$, we apply (13) to get $|f(k, x_k)| < c_k$ and $|f(k, y_k)| < c_k$ for all $k \in \mathbb{N}$. So we have

$$\sum_{k=1}^{N_1-1} |f(k, x_k)| < \sum_{k=1}^{N_1-1} c_k < \frac{\varepsilon}{3} \quad \text{and} \quad \sum_{k=1}^{N_1-1} |f(k, y_k)| < \sum_{k=1}^{N_1-1} c_k < \frac{\varepsilon}{3}. \quad (16)$$

Therefore, utilizing (15) and (16), we obtain

$$||P_f(y) - P_f(x)|| = \sum_{k=1}^{N_1-1} |f(k, y_k) - f(k, x_k)|$$

$$= \sum_{k=1}^{N_1-1} |f(k, y_k) - f(k, x_k)| + \sum_{k=1}^{N_1-1} |f(k, x_k) - f(k, x_k)|$$

$$\leq \sum_{k=1}^{N_1-1} |f(k, y_k) - f(k, x_k)| + \sum_{k=1}^{N_1-1} |f(k, y_k)| + \sum_{k=1}^{N_1-1} |f(k, x_k)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$
for each $k$, and $|x_k| < \infty$ such that $|f(k, x_k)| = 1$ for $k \leq i$. It follows from the assumption that for each $k \geq i$, $|f(k, x_k)| \leq c_k + \alpha k' l^k$. Therefore, $\sum_{k=i}^{\infty} |f(k, x_k)| \leq \sum_{k=i}^{\infty} c_k + \alpha \sum_{k=i}^{\infty} k' l^k$. Since $(x_k) \in F_i(p)$, we have $|f(k, x_k)| \leq |x_k|$. Thus, we have $|f(k, t)| \leq |x_k|$. Therefore, $\sum_{k=i}^{\infty} |f(k, t)| \leq \sum_{k=i}^{\infty} |x_k|$. Hence, $P_i$ acts from $F_i(p)$ to $\ell_i$.

(⇒) Suppose that $P_i$ acts from $F_i(p)$ to $\ell_i$. For each $\alpha, \beta > 0$ and for each positive integer $k$, we define

$$A(k, \alpha, \beta) = \left\{ t \in \mathbb{R} : |f(k, t)| \leq \min\left\{ \beta^u, \frac{|f(k, t)|}{\alpha k'^2} \right\} \right\}$$

and

$$B(k, \alpha, \beta) = \sup \{ |f(k, t)| ; t \in A(k, \alpha, \beta) \}.$$ 

If $k' l^k \leq \beta^u$ and $t \in A(k, \alpha, \beta)$ then $|f(k, t)| \leq B(k, \alpha, \beta)$. And if $k' l^k \leq \beta^u$ and $t \notin A(k, \alpha, \beta)$ then $|f(k, t)| \leq \alpha k' l^k$. Thus we have that $|f(k, t)| \leq B(k, \alpha, \beta) + \alpha k' l^k$ whenever $k' l^k \leq \beta^u$.

Next we shall show that $(B(k, \alpha, \beta))_{k=1}^{\infty} \subseteq \ell_i$ for some $\alpha, \beta > 0$. Suppose that for each $\alpha, \beta > 0$,

$$\sum_{k=1}^{\infty} B(k, \alpha, \beta) = \infty.$$ 

We get that for each integer $i \in \mathbb{N}$,

$$\sum_{k=i}^{\infty} B(k, 2^i, \frac{1}{2^i}) = \infty.$$ 

Then there is a sequence of positive integers $n_0 = 0 < n_1 < n_2 < ... < n_i < ...$ and the least positive integer $n_i$ such that $\sum_{k=n_{i-1}+1}^{n_i} B(k, 2^i, \frac{1}{2^i}) > 1$. Therefore for each $i \in \mathbb{N}$, there is $\varepsilon_i > 0$ such that

$$\sum_{k=n_{i-1}+1}^{n_i} B(k, 2^i, \frac{1}{2^i}) \varepsilon_i (n_i - n_{i-1}) > 1.$$ 

(17)

Let $i \in \mathbb{N}$ be fixed. Since $f$ satisfies the condition $A(2, 0) \leq B(k, 2^i, \frac{1}{2^i}) < \infty$ for all $k \in \mathbb{N}$ with $n_{i-1} + 1 \leq k \leq n_i$. It follows from the definition of $B(k, 2^i, \frac{1}{2^i})$ that for each $k \in \mathbb{N}$ with $n_{i-1} + 1 \leq k \leq n_i$, there exists $x_k = B(k, 2^i, \frac{1}{2^i})$ such that

$$|f(k, x_k)| > B(k, 2^i, \frac{1}{2^i}) - \varepsilon_i.$$ 

(18)

From (17) and (18), for each $i \in \mathbb{N}$,

$$\sum_{k=n_{i-1}+1}^{n_i} |f(k, x_k)| > \sum_{k=n_{i-1}+1}^{n_i} B(k, 2^i, \frac{1}{2^i}) - \sum_{k=n_{i-1}+1}^{n_i} \varepsilon_i = \sum_{k=n_{i-1}+1}^{n_i} B(k, 2^i, \frac{1}{2^i}) - \varepsilon_i (n_i - n_{i-1}) > 1.$$ 

Thus

$$\sum_{i=1}^{\infty} \sum_{k=n_{i-1}+1}^{n_i} |f(k, x_k)| = \sum_{k=1}^{\infty} |f(k, x_k)| = \infty,$$ 

and hence we have $(f(k, x_k))_{k=1}^{\infty} \subsetneq \ell_i$. As $x_k = B(k, 2^i, \frac{1}{2^i})$ for all $k \in \mathbb{N}$ with $n_{i-1} + 1 \leq k \leq n_i$, we obtain that for each $k \in \mathbb{N}$ with $n_{i-1} + 1 \leq k \leq n_i$,

$$|x_k|^p \leq \frac{1}{k^2} \left( \frac{1}{2^i} \right)^u \leq \frac{1}{k^2} \quad \text{and} \quad |x_k|^p \leq \frac{|f(k, x_k)|}{k^2}.$$ 

(19)

Since $n_i$ is the least positive integer such that $\sum_{k=n_{i-1}+1}^{n_i} B(k, 2^i, \frac{1}{2^i}) > 1$, we obtain $\sum_{k=n_{i-1}+1}^{n_i} B(k, 2^i, \frac{1}{2^i}) \leq 1$. It follows from (19),
\[
\sum_{k=r+1}^{i} k'|x_i|^p = \sum_{k=r+1}^{i} k'|x_i|^p + n'|x_n|^p \\
\leq \sum_{k=r+1}^{i} k' \left( \frac{1 |f(k, x_i)|}{2^{k'}} \right) + n' \left( \frac{1}{2^{k'}} \right) \\
\leq \sum_{k=r+1}^{i} B(k, \frac{1}{2^{k'}}) + \frac{1}{2} \leq 1 + \frac{1}{2} - \frac{2}{2}.
\]

Thus \( \sum_{k=r}^{i} k'|x_i|^p = \sum_{k=r+1}^{i} k'|x_i|^p \leq \sum_{k=2}^{i} \frac{2}{2} = 2 \) and consequently \((x_k) \) is in \( F_r(p) \). By the assumption we have that \( P f(x) = (f(k, x_k))_{k=1}^{\infty} \in l_1 \), which is a contradiction. Accordingly \( (B(k, \alpha, \beta))_{k=1}^{\infty} \in l_1 \).

For all \( k \in \mathbb{N} \), we put \( c_k = B(k, \alpha, \beta) \) then we have \((*)\).

Before we prove Theorem 9, we need the result that, if \( f : \mathbb{R} \rightarrow \mathbb{R} \) is locally bounded then \( f \) satisfies the condition A(2), which was proved by Tainchai (1990).

**Theorem 9**

Let \( f : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R} \). Suppose that \( P_f \) is a superposition operator which acts from \( F_r(p) \) to \( \ell_q \). Then \( P_f \) is locally bounded if and only if \( f \) satisfies the condition A(2).

**Proof.**

(\( \Leftarrow \)) Suppose that \( f \) satisfies the condition A(2). Let \( z = (z_i) \in F_r(p) \). We now prove that \( P_f \) is locally bounded at \( z \). Since \( f \) satisfies the condition A(2) and \( P_f \) acts from to \( F_r(p) \) to \( \ell_q \), it follows from Theorem 8, there are \( \alpha, \beta > 0 \) and a sequence \( (c_i) \in \ell_q \) such that for each \( k \in \mathbb{N} \),

\[
|f(k, t)| \leq c_k + \alpha k'|t|^p \quad \text{whenever} \quad k'|t|^p \leq \beta^n.
\]

Let \( \eta = \frac{\beta}{2} \) and \( (x_i) \in F_r(p) \) with \( ||x - z|| \leq \eta \).

Since \( \eta \leq \infty \), there is an \( i \in \mathbb{N} \) such that

\[
|f(k, x_k)| \leq c_k + \alpha k'|x_k|^p \quad \text{whenever} \quad k'|x_k|^p \leq \beta^n.
\]

(\( \Rightarrow \)) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be locally bounded. Then for each \( k \geq 1 \), \( |f(k, x_k)| \leq c_k + \alpha k'|x_k|^p \). Consequently,

\[
\sum_{k=r}^{i} |f(k, x_k)| \leq \sum_{k=r}^{i} c_k + \alpha \sum_{k=r}^{i} k'|x_k|^p \leq \|c_i\| + \alpha \beta^n.
\]

For each \( k \in \mathbb{N} \), let \( m_k = \sup_{t \in \mathbb{R}} |f(k, t)| \). As \( f \) is locally bounded, there is an \( \eta > 0 \) such that

\[
\sum_{k=r}^{i} |f(k, x_k)| \leq \sum_{k=r}^{i} m_k \leq \sum_{k=r}^{i} \beta^n \leq \beta^n.
\]
satisfies the condition A(2), we have that \( m_k < \infty \) for every \( k \in \mathbb{N} \). And since \( \|x - z\| \leq \eta \), \( \|x_k - z_k\| \leq \left( \frac{n^r}{k^r} \right)^{\frac{1}{r}} \) for every \( k \in \mathbb{N} \) and then

\[
\|f(k, x_k)\| \leq m_k \text{ for every } k \in \mathbb{N}.
\]  

(24)

It follows from (23) and (24),

\[
\|Pf(x)\| = \left( \sum_{k=1}^{\infty} \|f(k, x_k)\| \right)^{\frac{1}{r}} = \left( \sum_{k=1}^{\infty} m_k \right)^{\frac{1}{r}} + \|c\| + \alpha M
\]

and hence \( \|Pf(x) - Pf(y)\| \leq \|Pf(x)\| + \|Pf(y)\| \leq \|Pf(x)\| + \sum_{k=1}^{\infty} m_k + \|c\| + \alpha M \). We choose \( \gamma = \|Pf(x)\| + \sum_{k=1}^{\infty} m_k + \|c\| + \alpha M \), so we obtain \( \|Pf(x) - Pf(y)\| \leq \gamma \). The operator \( Pf \) is then locally bounded at \( z \).

(\( \Rightarrow \) Suppose that \( Pf \) acting from \( F_r(p) \) to \( \ell_1 \) is locally bounded. We are going to show that \( f \) satisfies the condition A(2) by proving that \( f(k, \ldots) \) is locally bounded for all \( k \in \mathbb{N} \). Now let \( x \in F_r(p) \) and \( b \in \mathbb{R} \). We define a sequence \( y = (y_n) \) with

\[
y_n = \begin{cases} b, & n = k \\ 0, & n \neq k \end{cases}
\]

Observe that \( y \in Pf(x) \) and by assumption, there exist \( \alpha, \beta > 0 \) such that

\[
\|Pf(x) - Pf(y)\| \leq \beta \text{ whenever } \|x - y\| \leq \alpha. \tag{25}
\]

Let \( a \in \mathbb{R} \) with \( |a - b| \leq \left( \frac{\alpha}{k^r} \right)^{\frac{1}{r}} \) and let \( x = (x_n) \) be a sequence with \( x_n = \begin{cases} a, & n = k \\ 0, & n \neq k \end{cases} \). We suddenly have \( x \in F_r(p) \) and \( \|x - y\| = \left( \sum_{k=1}^{\infty} n^r |x_n - y_n|^{r_n} \right)^{\frac{1}{r}} \). It follows from (25) that \( \|Pf(x) - Pf(y)\| \leq \beta \) and thus

\[
\|f(k, a) - f(k, b)\| \leq \sum_{k=1}^{\infty} |f(n, x_n) - f(n, y_n)| = \|Pf(x) - Pf(y)\| \leq \beta.
\]

Hence \( f(k, \ldots) \) is locally bounded at \( b \in \mathbb{R} \).

The following corollary is readily verified by employing Theorem 9.

**Corollary 10**

Let \( f : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R} \) satisfy the condition A(2). If the superposition operator \( Pf \) acting from \( F_r(p) \) to \( \ell_1 \) is bounded then \( f \) satisfies the condition A(2).

**Corollary 11**

Let \( f : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R} \). If the superposition operator \( Pf \) acting from \( F_r(p) \) to \( \ell_1 \) is bounded then \( f \) satisfies the condition A(2).

**Lemma 12**

Let \( f : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R} \) satisfy the condition A(2). If for each \( \beta > 0 \) there is an \( \alpha(\beta) > 0 \) such that for any finite sequence \( (x_k) \),

\[
\sum_{k=1}^{\infty} |f(k, x_k)| \leq \alpha(\beta) \text{ provided } \sum_{k=1}^{\infty} k^r |x_k|^{r_n} \leq \beta^r.
\]

Then there exists a sequence \( c(\beta) = (c_k(\beta)) \in \ell_1 \), with \( c_k(\beta) \geq 0 \) for all \( k \in \mathbb{N} \) and \( \|c(\beta)\| \leq \alpha(\beta) \) such that for each \( k \in \mathbb{N} \),
\[
\text{If } f(k, t) \leq c_i(\beta) + 2 \frac{\alpha(\beta)}{\beta^2} k^\varepsilon t^\delta \quad \text{whenever } k^\varepsilon t^\delta \leq \beta^m.
\]

**Proof.**

Let \( \beta > 0 \). By the assumption, there is \( \alpha(\beta) > 0 \) such that for any finite sequence \( (x_i) \),

\[
\sum_{k=1}^{\infty} |f(k, x_i)| \leq \alpha(\beta) \quad \text{provided } \sum_{k=1}^{\infty} k^\varepsilon x_i^\delta \leq \beta^m. \tag{26}
\]

For each \( k \in \mathbb{N} \), we define

\[
h_\delta(k, t) = \max \left\{ 0, |f(k, t)| - 2 \frac{\alpha(\beta)}{\beta^m} k^\varepsilon t^\delta \right\}
\]

and

\[
c_\varepsilon(\beta) = \sup \{ h_\delta(k, t) ; k^\varepsilon t^\delta \leq \beta^m \}.
\]

Let \( k \in \mathbb{N} \) and \( t \in \mathbb{R} \) be such that \( k^\varepsilon t^\delta \leq \beta^m \). If

\[
h_\delta(k, t) = 0, |f(k, t)| \leq c_\varepsilon(\beta) + 2 \frac{\alpha(\beta)}{\beta^m} k^\varepsilon t^\delta
\]

otherwise, \( h_\delta(k, t) = |f(k, t)| - 2 \frac{\alpha(\beta)}{\beta^m} k^\varepsilon t^\delta \). Therefore for each \( \beta > 0 \) and \( k \in \mathbb{N} \), \( |f(k, t)| \leq c_\varepsilon(\beta) + 2 \frac{\alpha(\beta)}{\beta^m} k^\varepsilon t^\delta \) whenever \( k^\varepsilon t^\delta \leq \beta^m \). Next we shall show that \( c(\beta) = (c_\varepsilon(\beta)) \in \ell_1 \quad \text{and} \quad \|c(\beta)\| \leq \alpha(\beta) \).

Since \( f \) satisfies the condition \( A(2) \), \( h_\delta(k, t) \) is bounded on every bounded subset of real numbers for all \( k \in \mathbb{N} \), so we have that \( 0 \leq c_\varepsilon(\beta) < \infty \) for all \( k \in \mathbb{N} \). By the definition of \( c_\varepsilon(\beta) \), for each \( \varepsilon > 0 \) there exists the sequence \( y = (y_k) \) with \( k^\varepsilon y_k^\delta \leq \beta^m \) for all \( k \in \mathbb{N} \) and

\[
c_\varepsilon(\beta) < h_\delta(k, y_k) + \frac{\varepsilon}{2}\alpha(\beta).
\]

Let the sequence \( y' \) be defined as follow:

\[
y'_k = \begin{cases} y_k, & h_\delta(k, t) > 0 \\ 0, & h_\delta(k, t) = 0 \end{cases}.
\]

For any \( m \in \mathbb{N} \), a finite sequence \( (m_i) \) with \( m_1 = 1 < m_2 < \ldots < m_s = m \) can be found such that

\[
\sum_{k=1}^{m_i} k^\varepsilon y_k^\delta = \sum_{k=m_i}^{m_{i+1} - 1} k^\varepsilon y_k^\delta + \ldots + \sum_{k=m_s - 1}^{m_s} k^\varepsilon y_k^\delta
\]

and

\[
0 \leq m_i k^\varepsilon y_i^\delta \leq \beta^m.
\]

Let \( z^{(i)} = \sum_{k=m_i}^{m_{i+1} - 1} k^\varepsilon y_k^\delta \) for all \( i \in \{1, 2, \ldots, s - 2\} \) and \( z^{(s-1)} = \sum_{k=m_s - 1}^{m_s} k^\varepsilon y_k^\delta \) for all \( i \in \{1, 2, \ldots, s - 2\} \) and \( z^{(s-1)} \) and \( z^{(s)} \) be defined as follow:

\[
z^{(i)} = \begin{cases} z^{(i-1)}, & \text{for } i \in \{1, 2, \ldots, s - 2\} \\ z^{(s-1)} + z^{(s)}, & \text{for } i = s - 1 \end{cases}.
\]

Consequently for each \( i \in \{1, 2, \ldots, s - 1\} \), \( z^{(i)} \in \Phi \) and \( \sum_{k=1}^{m_i} k^\varepsilon y_k^\delta \leq \beta^m \). From (26), we obtain that

\[
\sum_{k=1}^{m_i} |f(k, z^{(i)})| \leq \alpha(\beta) \quad \text{for all } i \in \{1, 2, \ldots, s - 2\}.
\]

Accordingly for each \( i \in \{1, 2, \ldots, s - 2\} \),

\[
\sum_{k=m_i}^{m_{i+1} - 1} |f(k, z^{(i)})| \leq \alpha(\beta) \quad \text{and}
\]

\[
\sum_{k=m_s - 1}^{m_s} |f(k, z^{(s)})| \leq \alpha(\beta).
\]

We define \( f_i(k, t) = f(k, t), h_i(k, t) = 0 \). It follows from (28),

\[
\sum_{k=m_i}^{m_{i+1} - 1} |f_i(k, z^{(i)})| \leq \alpha(\beta) \quad \text{for every } i \in \{1, 2, \ldots, s - 2\}.
\]

and

\[
\sum_{k=m_s - 1}^{m_s} |f_i(k, z^{(s)})| \leq \alpha(\beta).
\]

By definitions of \( y' \), \( f_i(k, t) \) and \( h_i(k, t) \) we obtain
\[
 h_j(k, y_j) = |f_r(k, y_j)| - 2 \frac{\alpha(\beta)}{\beta^a} k^j l^j
\]  

(31)

and hence it follows from (27), (29), (30) and (31),

\[
\sum_{k=1}^{n-1} c_k(\beta) < \sum_{k=1}^{n-1} h_j(k, y_j) + \sum_{k=1}^{n-1} \frac{\varepsilon}{2^k}
\]

\[
= \sum_{k=1}^{n-1} \left( |f_r(k, y_j)| - 2 \frac{\alpha(\beta)}{\beta^a} k^j l^j \right) + \sum_{k=1}^{n-1} \left( |f_r(k, y_j)| - 2 \frac{\alpha(\beta)}{\beta^a} k^j l^j \right) + \ldots +
\]

\[
\sum_{k=1}^{n-1} \left( |f_r(k, z^n_j)| - 2 \frac{\alpha(\beta)}{\beta^a} k^j l^j \right) + \frac{\varepsilon}{2^k}
\]

\[
\leq (s-1) \alpha(\beta) - 2 \frac{\alpha(\beta)}{\beta^a} \left( \sum_{k=1}^{n-1} k^j l^j \right) + \sum_{k=1}^{n-1} \frac{\varepsilon}{2^k}
\]

\[
\leq (s-1) \alpha(\beta) - 2 \frac{\alpha(\beta)}{\beta^a} (s-2) \frac{\beta^a}{a} + \sum_{k=1}^{n-1} \frac{\varepsilon}{2^k}
\]

\[
= \alpha(\beta) + \sum_{k=1}^{n-1} \frac{\varepsilon}{2^k}
\]

and thus \( \sum_{k=1}^{n-1} c_k(\beta) = \lim_{n \to \infty} \sum_{k=1}^{n-1} c_k(\beta) \leq \alpha(\beta) + \varepsilon \). Since \( \varepsilon > 0 \) is arbitrary, \( \sum_{k=1}^{n-1} c_k(\beta) \leq \alpha(\beta) \) and thus we get that \( \|c(\beta)\| \leq \alpha(\beta) \). Hence the lemma is proved.

**Theorem 13**

(\( \Leftarrow \)) Suppose that the condition holds. Let \( \rho > 0 \) and \( x \in F(\rho) \) with \( \|x\| \leq \rho \). Then \( \left( \sum_{k=1}^{n-1} k^j l^j \right)^{\frac{1}{2}} \leq \rho \) and consequently \( k^j l^j \leq \rho^a \) for every \( k \in N \).

By the assumption, there is an \( \alpha(\rho) > 0 \) and a sequence \( c(\rho) = (c_k(\rho)) \in \ell_1 \) such that for each \( k \in N \), \( |f_r(k, t)| \leq c_k(\rho) + 2 \frac{\alpha(\rho)}{\rho^a} k^j l^j \) whenever \( k^j l^j \leq \rho^a \).
Superposition operator on $E^r(p)$ and $F(p)$

Sama-ae, A.

Let $f : N \times \mathbb{R} \to \mathbb{R}$. The superposition operator $P_i$ acting from $F_i(p)$ to $\ell_i$ is continuous if and only if $f$ satisfies the condition $A(2)$.

Proof.

(⇒) Suppose that $P_i$ is continuous on $F_i(p)$. Let $k \in N$, $t_0 \in \mathbb{R}$, and $\varepsilon > 0$ be given. Since $P_i$ is continuous at $t_0 e^{i\alpha} \in F_i(p)$, there is a $\delta > 0$ such that for each $z = (z_n) \in F_i(p)$,

$$
\|P_i(z) - P_i(t_0 e^{i\alpha})\| < \varepsilon
$$

whenever $$
\|z - t_0 e^{i\alpha}\| < \delta.
$$

Let $t \in \mathbb{R}$ be such that $|t - t_0| < \frac{\delta^m}{\sqrt{k}}$ and $y_n = \begin{cases} 0, & n \neq k \\ t, & n = k \end{cases}$. Then $y = (y_n) \in F_i(p)$ and $\|y - t_0 e^{i\alpha}\| = \|k'|t - t_0 l^m\| < \frac{\delta^m}{\sqrt{k}}$. By (32), we have $\|P_i(y) - P_i(t_0 e^{i\alpha})\| < \varepsilon$ and thus $|f(k, t) - f(k, t_0)| < \frac{\delta^m}{\sqrt{k}}$.

Hence the function $f(k,\cdot)$ is continuous on $\mathbb{R}$ for all $k \in N$. That is $f$ satisfies the condition $A(2)$.

(⇐) Suppose that $f(k,\cdot)$ is continuous on $\mathbb{R}$ for all $k \in N$. We shall show that $P_i$ is continuous on $F_i(p)$. Let $x = (x_n) \in F_i(p)$ and $\varepsilon > 0$. Since $f$ satisfies the condition $A(2)$, so we get that $f$ satisfies the condition $A(2)$, so we get that $f$ satisfies the condition $A(2)$. As $P_i$ acts from $F_i(p)$ to $\ell_i$, so we apply Theorem 8, then there exist $\alpha, \beta > 0$ and a sequence $(c_k) \in \ell_i$ such that for each $k \in N$,

$$
|f(k, t)| \leq c_k + \alpha |k'|t|^\beta \quad \text{whenever} \quad |k'|t|^\beta \leq \beta^m.
$$

As $(x_n) \in F_i(p)$, we get $\sum k'|x_n|^\beta < \infty$ and therefore

$$
\lim_{k \to \infty} k'|x_n|^\beta = 0.
$$

And since $(c_k) \in \ell_i$, we obtain $\sum_{k \in N} c_k|z| < \infty$. Accordingly there is $N \in \mathbb{N} - \{1\}$ such that

$$
\|P_i(z) - P_i(t_0 e^{i\alpha})\| < \varepsilon
$$

whenever $\|z - t_0 e^{i\alpha}\| < \delta$.
\[ |x_k|^h < \frac{1}{k} \left( \frac{\beta}{2} \right)^{\frac{m}{h}} \] for all \( k \geq N_i \) and \( \sum_{k=N_i}^{\infty} c_k < \frac{\varepsilon}{6} \).  

(35)

Using (33), we have that \( |f(k, x_k)| \leq c_k + \alpha k |x_k|^h \) for all \( k \geq N_i \) and hence

\[ \sum_{k=N_i}^{\infty} |f(k, x_k)| \leq \sum_{k=N_i}^{\infty} c_k + \alpha \sum_{k=N_i}^{\infty} k |x_k|^h < \frac{\varepsilon}{6} + \frac{\alpha}{12} \left( \frac{\varepsilon}{\alpha} \right) < \frac{\varepsilon}{3}, \]  

(36)

Since \( f(k,.) \) is continuous at \( x_k \) for all \( k \in \{1, 2, \ldots, N_i - 1\} \), there exist \( \delta > 0 \) with

\[ \frac{1}{\min \left\{ \frac{\beta}{2}, \frac{1}{12} \left( \frac{\varepsilon}{\alpha} \right) \right\}} \]  

such that for each \( k \in \{1, 2, \ldots, N_i - 1\} \) and \( t \in \mathbb{R} \)

\[ |f(k, t) - f(k, x_k)| < \frac{\varepsilon}{3(N_i - 1)} \] whenever \( |t - x_k| < \delta \).  

(37)

Let \( z = (z_k) \in \text{Fr}(p) \) be such that \( ||x - z|| < \delta \) hence

\[ |z_k - x_k| < \delta k \left( \frac{\beta}{k} \right)^{\frac{m}{h}} \] for all \( k \in \mathbb{N} \).  

(38)

By (37), we get that \( |f(k, z_k) - f(k, x_k)| < \frac{\varepsilon}{3(N_i - 1)} \) for every \( k \in \{1, 2, \ldots, N_i - 1\} \) and thus

\[ \sum_{k=1}^{N_i} |f(k, z_k) - f(k, x_k)| < \frac{\varepsilon}{3}. \]  

(39)

By choosing \( \delta \leq \min \left\{ 1, \left( \frac{\beta}{2} \right)^{\frac{m}{h}}, \frac{1}{12} \left( \frac{\varepsilon}{\alpha} \right) \right\} \) and \( ||x - z|| < \delta \), we obtain \( \|(z_k)_{k=N_i}^{\infty} - (x_k)_{k=N_i}^{\infty}\| \leq ||z - x|| < \delta \).

(33) \( \sum_{k=N_i}^{\infty} k |x_k|^h \leq \frac{\varepsilon}{6} + \frac{\alpha}{12} \left( \frac{\varepsilon}{\alpha} \right) < \frac{\varepsilon}{3} \),

\[ \left( \sum_{k=N_i}^{\infty} k |x_k|^h \right)^{\frac{1}{h}} = \|(z_k)_{k=N_i}^{\infty}\| \leq \|(x_k)_{k=N_i}^{\infty}\| + \|(z_k)_{k=N_i}^{\infty} - (x_k)_{k=N_i}^{\infty}\| < \left( \frac{1}{12} \left( \frac{\varepsilon}{\alpha} \right) \right)^{\frac{1}{h}} + \left( \frac{1}{12} \left( \frac{\varepsilon}{\alpha} \right) \right)^{\frac{1}{h}} = \left( \frac{1}{6} \right)^{\frac{1}{h}}, \]  

this implies \( \sum_{k=N_i}^{\infty} k |x_k|^h \leq \left( \frac{1}{6} \right)^{\frac{1}{h}} \frac{\varepsilon}{\alpha} \leq \frac{\varepsilon}{6\alpha} \). For each \( k \geq N_i \), by utilizing (35) and (38),

\[ |z_k| \leq |x_k| + |z_k - x_k| < \left( \frac{\beta}{2^m k^h} \right)^{\frac{1}{h}} + \left( \frac{\beta}{k} \right)^{\frac{1}{h}} \leq \frac{1}{2} \left( \frac{\beta}{k} \right)^{\frac{1}{h}} + \frac{1}{2} \left( \frac{\beta}{k} \right)^{\frac{1}{h}} = \left( \frac{\beta}{k} \right)^{\frac{1}{h}}, \]  

and hence \( k |z_k|^h \leq \beta^m \). It follows from (33), \( |f(k, z_k)| \leq c_k + \alpha k |z_k|^h \) for all \( k \geq N_i \) and then
Superposition operator on $E_r(p)$ and $F_r(p)$

Sama-ae, A.

Vol. 24 No. 3 Jul.-Sep. 2002

\[ \sum_{k=N_0}^{\infty} |f(k, z_k)| \leq \sum_{k=N_0}^{\infty} c_i + \alpha \sum_{k=N_0}^{\infty} k' |z_k|^n < \frac{\varepsilon}{6} + \alpha \frac{\varepsilon}{6\alpha} = \frac{\varepsilon}{3}. \]  

(40)

By using (36), (39) and (40), we have

\[
\|P_f(z) - P_f(x)\| = \sum_{k=1}^{\infty} |f(k, z_k) - f(k, x_k)| \\
= \sum_{k=1}^{N_0} |f(k, z_k) - f(k, x_k)| + \sum_{k=N_0}^{\infty} |f(k, z_k) - f(k, x_k)| \\
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\]

The proof of this theorem is then completed.

The last of our results is Corollary 15 which follows from Theorem 9 and 14.

**Corollary 15**

Let $f : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R}$. If the superposition operator $P_f$ acting from $F_r(p)$ to $\ell_1$ is continuous then $P_f$ is locally bounded.

**References**


