Further results on (1,0,0)-F-Face magic mean graphs

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Abstract

A (1,0,0)-F-Face magic mean labeling is an assignment of labels to the vertices of planar graph such that the mean weight of each face including an excellent survey of graph labeling can be found in Gallian (2005). Many kinds of labelings have been studied and an injection of labels to the vertices of planar graph including the edges. Planar graph is a graph which can be drawn in a plane so that no two edges intersect. Throughout this paper, a graph $G$ contains a finite, connected, undirected planar graph having neither loops nor multiple edges. A planar graph is a graph which can be drawn in a plane so that no two edges intersect. Throughout this paper, a graph $G$ contains a finite, connected, undirected planar graph having neither loops nor multiple edges. A planar graph is a graph which can be drawn in a plane so that no two edges intersect.

A globe graph is a graph produced by joining two isolated vertices by $n$ paths of length two. It is isomorphic to $K_{2,n}$. A wheel graph consists of a rim and spokes. The rim edges are the edges of a cycle and the spokes are edges connecting the central vertex with each node of the rim.

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1. Introduction

A function $\psi$ on a graph $G$ is called a mean labeling of graph $G$ if $\psi: V(G) \rightarrow \{0,1,2,\ldots,|E(G)|\}$ is injective and the induced edge function $\psi^*: E(G) \rightarrow \{1,2,\ldots,|E(G)|\}$ defined as follows is bijective.

For each edge $uv$, $\psi^*(uv) = \begin{cases} \frac{\psi(u) + \psi(v)}{2} & \text{if } \psi(u) + \psi(v) \text{ is even} \\ \frac{\psi(u) + \psi(v) + 1}{2} & \text{if } \psi(u) + \psi(v) \text{ is odd} \end{cases}$

The graph which admits mean labeling is called a mean graph (Durai Baskar & Arockiaraj, 2016).
A graph \( G \) is magic if the edges of \( G \) can be labeled by the numbers \( 1,2,3,\ldots,|E(G)| \), so that the sum of the labels of all the edges incident with any vertex is the same (Hartsfield & Ringel, 1990).

A bijection \( \phi: E(G) \rightarrow \{1,2,\ldots,|E(G)|\} \) is called a (1,0,0)-F-Face magic mean labeling of \( G \) if the induced face labeling

\[
\phi^*(f_i) = \left[ \frac{\text{sum of the labels of the vertices in the boundary of } f_i}{\deg(f_i)} \right] = \frac{\sum_{v \in F(f_i)} \phi(v)}{\deg(f_i)} = k, \text{ a constant}
\]

for each face \( f_i \), including the exterior face of \( G \), where \( \deg(f_i) \) is the degree of the face \( f_i \), that is the number of edges that bound the face. The graph which admits F-Face magic mean labeling is called a F-Face magic mean graph (Arockiaraj & Meena Kumari, 2017). Magic meanness property of slanting ladder has been discussed by Meena Kumari and Arockiaraj (2017). Arockiaraj and Meena Kumari (2018) investigated the F-face magic mean labeling for the vertex duplication of cycle \( C_n \). (Durai Baskar et al., 2016) discussed the geometric meanness property of the grid graph \( P_m \times P_n \).

Motivated by these works, the authors of this paper introduce the (1,0,0)-F-Face magic mean labeling of graphs as follows:

2. Main Results

**Theorem 2.1.** The graph \( \overline{C}_n \) is a (1,0,0)-F-Face magic mean graph with face constant \( n \) if and only if \( n \) is even.

**Proof.** If \( n \) is odd, \( \overline{C}_n \) is a nonplanar graph.

Assume that \( n \) is even.

Let \( \{v_i: 1 \leq i \leq n\} \) be the vertices of \( C_n \) and \( u_i \) be the respective duplicating vertex of \( v_i \), for \( 1 \leq i \leq n \).

Then \( E(\overline{C}_n) = \{v_iv_{i+1}; 1 \leq i \leq n - 1\} \cup \{u_iv_{i+1}; 2 \leq i \leq n\} \cup \{u_iv_{i+1}; 1 \leq i \leq n - 1\} \cup \{u_nv_n, u_nv_1, v_nv_1\} \) and

\[
F(\overline{C}_n) = \{f_1 = (v_{i-1}v_i, v_{i+1}v_i, u_1u_{i+1}, u_1u_{i-1}); 2 \leq i \leq n - 1\} \cup \{f_2 = (v_nv_n, v_nu_n, u_nv_n)\}
\]

Then \( f_1 = \left( \bigcup_{i \equiv 0 \pmod{2}} v_{i+1}u_i, \bigcup_{i \equiv 0 \pmod{2}} v_{i+1}u_i \right) \) and \( f_0 = \left( \bigcup_{i \equiv 1 \pmod{2}} v_{i+1}u_i, \bigcup_{i \equiv 1 \pmod{2}} v_{i+1}u_i \right) \).

In \( \overline{C}_n \), \( |V(\overline{C}_n)| = 2n \) and \( |F(\overline{C}_n)| = n + 2 \).
Define \( \phi: V(\hat{C}_n) \to \{1,2,...,2n\} \) as follows: For \( 1 \leq i \leq n \), \( \phi(v_i) = \begin{cases} n + 1, & i = 1 \\ 2n - i + 2, & i \equiv 1(\text{mod}2), i \neq 1 \end{cases} \)

\[
\phi(u_i) = \begin{cases} n, & i = 2 \\ 2n - i + 1, & i \equiv 1(\text{mod}2). \end{cases}
\]

Then the induced face labeling \( \phi^* \) in \( G \) is obtained as follows:

\[
\phi^*(f_1) = \frac{1}{4} \left( \phi(v_{i-1}) + \phi(v_i) + \phi(v_{i+1}) + \phi(u_i) \right)
\]

for all \( 3 \leq i \leq n - 1, i \equiv 0(\text{mod}2) \)

\[
\phi^*(f_2) = \frac{1}{4} \left( \phi(v_1) + \phi(v_2) + \phi(v_{n}) \right) = n,
\]

\[
\phi^*(f_3) = \frac{1}{4} \left( \phi(v_{i-1}) + \phi(v_i) + \phi(v_i) + \phi(u_i) \right)
\]

for all \( 4 \leq i \leq n, i \equiv 0(\text{mod}2) \) and

\[
\phi^*(f_0) = \frac{1}{4} \left( \sum_{i=1}^{n-1} \phi(u_i) + \sum_{i=2}^{n} \phi(v_i) \right) = n.
\]

Thus \( \phi^* (f) = n \), for each face \( f \) of \( G \).

So, \( G \cong \hat{C}_n \) is a \((1,0,0)\)-F-face magic mean graph with face constant \( n \) if \( n \) is even. A \((1,0,0)\)-F-face magic mean labeling of \( \hat{C}_n \) is shown in Figure 1.

**Theorem 2.2.** The graph \( G \) obtained by subdividing the rim edges of a wheel graph \( W_n = C_n + K_1 \) by a vertex is a \((1,0,0)\)-F-Face magic mean graph for \( n \geq 3 \) with face constant \( n \).

**Proof.** Let \( V(G) = \{u_i; 1 \leq i \leq n\} \cup \{v_i; 1 \leq i \leq n\} \cup \{u\}, \)

\[
E(G) = \{uu_i; 1 \leq i \leq n\} \cup \{uv_i; 1 \leq i \leq n\} \cup \{vu_{i+1}; 1 \leq i \leq n - 1\} \cup \{vu_1\} \quad \text{and}
\]

\[
F(G) = \{f_1 = (uu_i, u_i v_i, v_{i+1}u_i, u_{i+1}u); 1 \leq i \leq n - 1\}
\]

\[
\cup \{f_n = (uu_n, u_n v_n, v_1 u_1, u_1u)\} \cup \{f_0 = \left( \bigcup_{i=1}^{n} u_i v_i, \bigcup_{i=2}^{n-1} v_i u_{i+1}, v_{n} u_{n} \right) \}
\]

![Figure 1](image URL)  

A \((1,0,0)\)-F-Face magic mean labeling of \( \hat{C}_n \) with face constant 8.
where $f_0$ is the outer face of $G$, $\text{deg}(f_i) = 4, 1 \leq i \leq n$ and $\text{deg}(f_0) = 2n$, $|V(G)| = 2n + 1$.

Case (i) $n$ is odd.

Define $\phi: V(G) \to \{1, 2, \ldots, 2n + 1\}$ as follows:
\[
\phi(u) = (3n - 3)/2,
\]
\[
\phi(u_i) = \begin{cases} 
\frac{i+1}{2}, & i \equiv 1 \,(\text{mod} 2) \\
\frac{n+i+1}{2}, & i \equiv 0 \,(\text{mod} 2) \
\end{cases}
\]
\[
\phi(v_i) = \begin{cases} 
2n - i + 1, & 1 \leq i \leq \frac{n+3}{2} \\
2n - i, & \frac{n+5}{2} \leq i \leq n - 1 \\
2n + 1, & i = n.
\end{cases}
\]

Then the induced face labeling $\phi^*$ on $G$ is obtained as $\phi^*(f) = n$ for each face $f$ of $G$. Thus $\phi$ is a $(1,0,0)$-F-Face magic mean labeling of $G$.

Case (ii) $n$ is even

Define $\phi: V(G) \to \{1, 2, \ldots, 2n + 1\}$ as follows:
\[
\phi(u) = 3n/2,
\]
\[
\phi(u_i) = \begin{cases} 
\frac{i+1}{2}, & i \equiv 1 \,(\text{mod} 2) \\
\frac{n+i+1}{2}, & i \equiv 0 \,(\text{mod} 2) \
\end{cases}
\]
\[
\phi(v_i) = \begin{cases} 
2n - i + 2, & 1 \leq i \leq \frac{n}{2} - 1 \\
2n - \frac{n}{2} + 1, & i = \frac{n}{2} \\
2n - i, & \frac{n}{2} + 1 \leq i \leq n - 1 \\
2n - \frac{n}{2} + 2, & i = n
\end{cases}
\]

Then the induced face labeling $\phi^*$ on $G$ is obtained as $\phi^*(f) = n$ for each face $f$ of $G$. Thus $\phi$ is a $(1,0,0)$-F-Face magic mean labeling of $G$. A $(1,0,0)$-F-Face magic mean labeling of $C_8 + K_1$ and $C_7 + K_1$ are shown in Figure 2.

Figure 2. A $(1,0,0)$-F-Face magic mean labeling of $C_8 + K_1$ and $C_7 + K_1$ having face mean constant 8 and 7 respectively.
Theorem 2.3. The globe graph $K_{2,n}$, $n \geq 1$ is a $(1,0,0)$-face magic mean graph.

Proof. Let $V(G) = \{u_i; 1 \leq i \leq 2\} \cup \{v_i; 1 \leq i \leq n\}$,  
$E(G) = \{u_1v_i; 1 \leq i \leq n\} \cup \{u_2v_i; 1 \leq i \leq n\}$ and  
$F(G) = \{f_i = (u_1v_i, u_i v_{i+1}, u_2v_i, u_2v_{i+1}); 1 \leq i \leq n - 1\} \cup \{f_0 = (u_1v_1, u_1v_n, u_2v_1, u_2v_n)\}$

where $f_0$ is the outer face of $G$, $\text{deg}(f_i) = 4, 0 \leq i \leq n, |V(G)| = 2n + 1$.

Case (i) $n$ is even.

Define $\phi: V(G) \rightarrow \{1,2,3,\ldots,2n+1\}$ as follows:

For $i \equiv 1(\text{mod}2), \phi(v_i) = \begin{cases}  
\frac{i+1}{2}, & 1 \leq i \leq 5 \\
\frac{i+3}{2}, & 1 \leq i \leq 5 \\
n, & 1 \leq i \leq 5 \end{cases}$

For $i \equiv 0(\text{mod}2), \phi(v_i) = \begin{cases}  
n, & i = 11 \\
\frac{i-11}{2}, & 13 \leq i \leq n-3 \\
\frac{n-i}{2}, & 2 \leq i \leq 4 \\
\frac{n-i}{2}, & 6 \leq i \leq 8 \\
n, & i = n-2 \\
\frac{n-i}{2}, & 10 \leq i \leq n-4 \\
n, & i = n \\
\frac{n}{2}, & i = n \end{cases}$

Then the induced face labeling $\phi^*$ on $G$ is obtained as $\phi^*(f_i) = \frac{n}{2} + 1$ for each face $f_i$ of $G$. Thus $\phi$ is a $(1,0,0)$-Face magic mean labeling of $G$.

Case (ii) $n \equiv 1(\text{mod}4)$

Assume that $n \geq 5$. Define $\phi: V(G) \rightarrow \{1,2,3,\ldots,2n+1\}$ as follows:

For $1 \leq i \leq n, i \neq n-2, \phi(v_i) = \begin{cases}  
\frac{i+1}{2}, & i \equiv 1(\text{mod}2) \\
n-i, & i \equiv 0(\text{mod}2) \\
n, & i = n-2 \\
n, & i = n-1 \\
\frac{n}{2}, & i = n \\
n, & i = n \end{cases}$

Then the induced face labeling $\phi^*$ on $G$ is obtained as $\phi^*(f_i) = \frac{3n+3}{4}$ for each face $f_i$ of $G$. Thus $\phi$ is a $(1,0,0)$-Face magic mean labeling of $G$. 
Case (iii) $n \equiv 3 \pmod{4}$

Assume that $n \geq 3$. Define $\phi: V(G) \to \{1, 2, 3, \ldots, 2n + 1\}$ as follows:

For $1 \leq i \leq n$ and $i \neq n - 2$,

$$\phi(v_i) = \begin{cases} 
  i + 1, & i \equiv 1 \pmod{2} \\
  n - i, & i \equiv 0 \pmod{2}.
\end{cases}$$

$$\phi(v_{n-2}) = n,$$

$$\phi(u_1) = n - 1$$

and

$$\phi(u_2) = n + 2.$$

Then the induced face labeling $\phi^*$ on $G$ is obtained as $\phi^*(f) = \frac{3n+1}{4}$ for each face $f$ of $G$. Thus $\phi$ is a $(1,0,0)$-Face magic mean labeling of $G$. When $n = 1$, the graph $K_{2,1}$ is a tree and so it admits a $(1,0,0)$-Face magic mean labeling. A $(1,0,0)$-Face magic mean labeling of $K_{2,10}$, $K_{2,7}$ and $K_{2,9}$ are shown in Figures 3, 4 and 5 respectively.

**Theorem 2.4.** Let $G$ be a graph obtained from cycle $C_n$ by identifying each of its edge with an edge of a copy of $C_4$. Then $G$ is a $(1,0,0)$-F-face magic mean graph.

**Proof.** Let the vertex set of $C_n$ be $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and the vertex set of $i^{th}$ copy of $C_4$ be $V(C_4) = \{u_{i1}, u_{i2}, u_{i3}, u_{i4}\}$. Let the edge $v_i v_{i+1}$ of cycle $C_n$ be identified with edge $u_{i1}u_{i2}$ of $i^{th}$ copy of $C_4$ and the edge $v_nv_1$ be identified with the edge $u_{i1}u_{i4}$.

![Figure 3](image1.png)  

**Figure 3.** A $(1,0,0)$-F-face magic mean labeling of $K_{2,10}$ with face mean constant 7.

![Figure 4](image2.png)  

**Figure 4.** A $(1,0,0)$-F-face magic mean labeling of $K_{2,7}$ with face mean constant 6.
\( u_{n1}u_{n2} \) of \( n^3 \) copy of \( C_4 \). Let \( F(G) \{ f_i = (v_iu_{i+1}, v_iu_{i+2}, u_{i+2}u_{i+3}, u_{i+4}v_{i+3}) : 1 \leq i \leq n - 1 \} \cup \{ f_n = (v_n, v_1u_{n4}, u_{n3}u_{n4}, u_{n3}v_n) \} \cup \{ f_o = (U_{i=1}^n v_iu_{i+3}u_{i+4}u_{i+3}u_{i+4}) \} \cup \{ f_i = (U_{i=1}^{n-1} v_iu_{i+1}, v_nv_i) \} \) where \( \text{deg}(f_i) = 4, 1 \leq i \leq n, \text{deg}(f_o) = 3n \) and \( \text{deg}(f_i) = n \).

Define \( \phi : V(G) \rightarrow \{ 1, 2, 3, \ldots, 3n \} \) as follows:

For \( 1 \leq i \leq n, \phi(v_i) = n + i \),

\[
\phi(u_{i3}) = n - i + 1 \quad \text{and} \quad \phi(u_{i4}) = \begin{cases} 
3n - i, & 1 \leq i \leq n - 1 \\
3n, & i = n
\end{cases}
\]

Then the induced face labeling \( \phi^* \) on \( G \) is obtained as

\[
\phi^*(f_i) = \begin{cases} 
\frac{3n + 1}{2}, & n \equiv 0 \text{ (mod 2)} \\
\frac{3n}{2}, & n \equiv 1 \text{ (mod 2)}
\end{cases}
\]

Thus \( \phi \) is a \((1,0,0)\)-Face magic mean labeling of \( G \). The \((1,0,0)\)-Face magic mean labeling of \( G_5 \) obtained by identifying each edge with an edge of a copy of \( C_4 \) is shown in Figure 6.

**Theorem 2.5.** Slanting ladder \( SL_{mn} \), \( n \geq 2 \) is a \((1,0,0)\)-Face magic mean graph with face mean constant \( n \).

**Proof.** Let \( G = SL_{mn}, \) where \( V(G) = \{ u_i : 1 \leq i \leq n \} \cup \{ v_i : 1 \leq i \leq n \} \),

\( E(G) = \{ u_iu_{i+1} : 1 \leq i \leq n - 1 \} \cup \{ v_iu_{i+1} : 1 \leq i \leq n - 1 \} \cup \{ u_iv_{i+2} : 2 \leq i \leq n \} \) and

\( F(G) = \{ f_i = (u_iu_{i+2}, v_iu_{i+1}, u_{i+1}v_i, u_{i+2}v_{i+1}) : 1 \leq i \leq n - 2 \} \cup \{ f_o = (\bigcup_{i=1}^{n-1} u_iu_{i+1}, \bigcup_{i=1}^{n-1} v_iv_{i+1}, u_{n-2}v_{n-1}) \} \)

in which \( f_o \) is the outer face of \( G \).
In $G, |V(G)| = 2n, \ deg(f_i) = 4, 1 \leq i \leq n - 2$ and $\ deg(f_0) = 2n$.

Case (i) $n$ is odd.

Define $\phi: V(G) \rightarrow \{1, 2, \ldots, 2n\}$ as follows:

$$\phi(u_i) = \begin{cases} \frac{2n+i}{2}, & i \equiv 1 \text{ (mod 2)} \\ \frac{n+i}{2}, & i \equiv 0 \text{ (mod 2)} \end{cases}$$

$$\phi(v_i) = \begin{cases} \frac{2n+i-1}{2}, & i \equiv 1 \text{ (mod 2)} \\ \frac{n-i}{2}, & i \equiv 0 \text{ (mod 2)}. \end{cases}$$

Then the induced face labeling $\phi^*$ on $G$ is obtained as $\phi^*(f_i) = n$, for all $1 \leq i \leq n - 2$ and $\phi^*(f_0) = \left\lfloor \frac{2n+1}{2} \right\rfloor = n$.

Case (ii) $n$ is even.

When $n = 2, SL_n$ is a tree. So it is a (1,0,0)-F-face magic mean graph.

For $n \geq 2$, Define $\phi: V(G) \rightarrow \{1, 2, \ldots, 2n\}$ as follows:

$$\phi(u_i) = \begin{cases} \frac{n+i+1}{2}, & i \equiv 1 \text{ (mod 2)} \\ \frac{2n+i}{2}, & i \equiv 0 \text{ (mod 2)} \end{cases}$$

$$\phi(v_i) = \begin{cases} \frac{2n+i-1}{2}, & i \equiv 1 \text{ (mod 2)} \\ \frac{n+i}{2}, & i \equiv 0 \text{ (mod 2)}. \end{cases}$$

Then the induced face labeling $\phi^*$ on $G$ is obtained as $\phi^*(f_i) = n$, for all $1 \leq i \leq n - 2$ and $\phi^*(f_0) = \left\lfloor \frac{2n+1}{2} \right\rfloor = n$.

Thus $\phi$ is a (1,0,0)-F-Face magic mean labeling of $SL_n, n \geq 2$ with face mean constant $n$. A (1,0,0)-F-Face magic mean labeling of $SL_7$ and $SL_8$ are shown in Figure 7.

**Theorem 2.6.** The grid graph $P_m \times P_n, m, n \geq 1$ is a (1,0,0)-F-face magic mean graph with face constant $\frac{mn}{2}$ if either $m$ or $n$ is even or both are even and $\frac{mn+1}{2}$ if both $m$ and $n$ are odd.
Let $G = P_m \times P_n$. Then $V(G) = \{u_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$,

$$E(G) = \{u_{ij}u_{i(j+1)}; 1 \leq i \leq m - 1, 1 \leq j \leq n\} \cup \{u_{ij}u_{i(j+1)}; 1 \leq i \leq m, 1 \leq j \leq n - 1\}$$

$$F(G) = \{f_{ij} = (u_{ij}u_{i(j+1)}u_{i(j+1)(j+1)}u_{i(j+1)(j+1)}u_{i(j+1)(j+1)}u_{i(j+1)(j+1)}); 1 \leq i \leq m - 1, 1 \leq j \leq n - 1\}$$

$$\bigcup \{f_0 = \sum_{j=1}^{m-1} u_{1j}u_{1(j+1)}u_{m(j+1)}u_{m(j+1)}u_{m(j+1)}u_{m(j+1)}\}$$

where $f_0$ is the outer face of $G$, $deg(f_{ij}) = 4, 1 \leq i \leq m - 1, 1 \leq j \leq n - 1$, $|V(G)| = 2n + 1$.

There are $(m - 1)(n - 1)$ faces of degree 4 and outer face is of degree $2m + 2n - 4$.

$$\phi(u_{11}) = 1,$$

$$\phi(u_{12}) = mn,$$

For $3 \leq i \leq n, \phi(u_{ij}) = \begin{cases} \phi(u_{ij-2}) + j - 2, & j \equiv 1(mod 2) \\ \phi(u_{ij-2}) - j + 2, & j \equiv 0(mod 2) \end{cases}$$

$$\phi(u_{2n}) = \phi(v_{1n-1}) - n + 1$$ and

For $2 \leq i \leq m, 1 \leq j \leq n - 1$,

$$\phi(u_{ij}) = \begin{cases} \phi(u_{i-1(j+1)}) - 1, & i + j \equiv 1(mod 2) \\ \phi(u_{i-1(j+1)}) + 1, & i + j \equiv 0(mod 2) \end{cases}$$

Then the induced face labeling $\phi^*$ on $G$ is obtained as $\phi^*(f) = mn/2$, for all faces $f$ of $G$. Thus $\phi$ is a $(1,0,0)$-F-Face magic mean labeling of $P_m \times P_n$ with face mean constant $mn/2$. A $(1,0,0)$-F-Face magic mean labeling of $P_6 \times P_7$ is shown in figure 8.

References


