On the class of fuzzy number sequences $b_{yp}^F$

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Abstract

We introduce the notion of $p$-bounded variation of fuzzy real number sequences, $b_{yp}^F$, for $1 \leq p < \infty$. We study some of its properties like completeness, monotonicity, convergence free and symmetricity. Also, we prove some inclusion results.

Keywords: fuzzy real number, solid space, symmetric space, convergence free

1. Introduction

The concept of fuzzy set, a set whose boundary is not sharp or precise was introduced by L. A. Zadeh in 1965. It is the origin of the new theory of uncertainty, distinct from the notion of probability. After the introduction of fuzzy sets, the scope for studies in different branches of pure and applied mathematics increased widely. The notion of fuzzy sets was successfully applied in studying sequence spaces with classical metrics, for example by Altinok, Et and Colak (2014), Altinok and Et (2015), Das (2014a, 2014b, 2017a, 2017b), Esi (2006, 2010a, 2010b, 2013), Et, Altinok, and Altin (2013), Nanda (1989), Tripathy and Dutta (2007), Tripathy and Das (2012), and Tripathy and Goswami (2015). Work with the concept of fuzzy metrics was done by Kelava and Seikkala (1984), Das (2014), Tripathy, Paul and Das (2015). In the field of fuzzy topology, some work was done by Dutta and Tripathy (2017), Tripathy and Debnath (2015), and Tripathy and Ray (2012).

2. Definitions and Preliminaries

Let $D$ denote the set of all closed and bounded intervals $X = [a_1, a_2]$ on $R$, the real line. For $X, Y \in D$ we define

$$X \leq Y, \text{ if } a_1 \leq b_1 \text{ and } a_2 \leq b_2,$$

$$d(X, Y) = \max \left( |a_1 - b_1|, |a_2 - b_2| \right),$$

where $X = [a_1, a_2]$ and $Y = [b_1, b_2]$. It is known that $(D, d)$ is a complete metric space. Also $\leq$ is a partial order in $D$.

Definition 2.1 A fuzzy real number $X$ is a fuzzy set on $R$, i.e. a mapping $X: R \to I = [0, 1]$ associating each real number $t$ with its grade of membership $X(t)$.

Definition 2.2 A fuzzy real number $X$ is called convex if $X(t) \geq X(s) \wedge X(r) = \min \{X(s), X(r)\}$, where $s < t < r$.

Definition 2.3. If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number $X$ is called normal.

Definition 2.4 A fuzzy real number $X$ is said to be upper-semi continuous if, for each $e > 0$, $X^{-1}([0, a + e])$, for all $a \in I$ is open in the usual topology of $R$. 

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The set of all upper-semi continuous, normal, convex fuzzy real numbers is denoted by $R(I)$. Throughout the article, by a fuzzy real number we mean that the number belongs to $R(I)$.

**Definition 2.5.** The $\alpha$-level set $[X]^{\alpha}$ of the fuzzy real number $X$, for $0 < \alpha \leq 1$, is defined by $[X]^{\alpha} = \{ t \in R: X(t) \geq \alpha \}$. If $\alpha = 0$, then it is the closure of the strong 0-cut. (The strong $\alpha$-cut of the fuzzy real number $X$, for $0 \leq \alpha \leq 1$ is the set $\{ t \in R: X(t) > \alpha \}$).

By 0-cut or 0-level set of the fuzzy real number $X$, we mean the closure of the strong 0-cut.

Throughout $\alpha$ means, $\alpha \in (0,1]$ unless stated otherwise.

For $X,Y \in R(I)$ consider a partial ordering $\leq$ as $X \leq Y$ if and only if $a^0_\alpha \leq a^0_\alpha$ and $b^0_\alpha \leq b^0_\alpha$, for all $\alpha \in (0,1]$, where $[X]^{\alpha} = [a^\alpha_\alpha, b^\alpha_\alpha]$ and $[Y]^{\alpha} = [a^\alpha_\alpha, b^\alpha_\alpha]$.

The arithmetic operations for $\alpha$-level sets are defined as follows:

Let $X,Y \in R(I)$ and the $\alpha$-level sets be $[X]^{\alpha} = [a^\alpha_\alpha, b^\alpha_\alpha]$, $[Y]^{\alpha} = [a^\alpha_\alpha, b^\alpha_\alpha]$, for all $\alpha \in (0,1]$. Then

$$[X \oplus Y]^{\alpha} = [a^\alpha_\alpha + a^\alpha_\alpha, b^\alpha_\alpha + b^\alpha_\alpha],$$

$$[X - Y]^{\alpha} = [a^\alpha_\alpha - b^\alpha_\alpha, b^\alpha_\alpha - a^\alpha_\alpha],$$

$$[X \odot Y]^{\alpha} = \left[\min_{i,j=0,1} a^\alpha_i b^\alpha_j, \max_{i,j=0,1} a^\alpha_i b^\alpha_j\right],$$

and $[Y]^{\alpha} = \left[\frac{1}{b^\alpha_\alpha}, \frac{1}{a^\alpha_\alpha}\right]$, $\alpha \in (0,1]$.

The set $R$ of all real numbers can be embedded in $R(I)$. For $r \in R$, $r \in R(I)$ is defined by

$$r(t) = \begin{cases} 1, & \text{for } t = r, \\ 0, & \text{for } t \neq r. \end{cases}$$

For $r \in R$ and $X \in R(I)$, the scalar product $rX$ is defined by

$$rX(t) = \begin{cases} X(r^{-1}t), & \text{if } r \neq 0, \\ 0, & \text{if } r = 0. \end{cases}$$

**Definition 2.6** The absolute value, $|X|$ of $X \in R(I)$ is defined by (Kaleva and Seikkala [1984])

$$|X|(t) = \begin{cases} \max(X(t),X(-t)), & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

A fuzzy real number $X$ is called non-negative if $X(t) = 0$, for all $t < 0$. The set of all non-negative fuzzy real numbers is denoted by $R^+(I)$. The additive identity and multiplicative identity in $R(I)$ are denoted by $\overline{0}$ and $\overline{1}$ respectively.

Let $\overline{d} : R(I) \times R(I) \to R$ be defined by

$$\overline{d}(X,Y) = \sup_{0 \leq \alpha \leq 1} d([X]^{\alpha}, [Y]^{\alpha}).$$

Then $\overline{d}$ defines a metric on $R(I)$.

**Definition 2.7** A fuzzy real numbers sequence $(X_k)$ is said to be level convergent to the fuzzy real number $X$ if, for each $\alpha \in (0,1]$,

$$\lim_{k \to \infty} a^\alpha_k = a^\alpha \quad \text{and} \quad \lim_{k \to \infty} b^\alpha_k = b^\alpha,$$

where $[X_k]^{\alpha} = [a^\alpha_k, b^\alpha_k]$, for all $k \in N$ and $[X]^{\alpha} = [a^\alpha, b^\alpha]$. If the convergence is uniform in $\alpha$, then we say that $(X_k)$ converges uniformly to $X$.

**Definition 2.8** A sequence $(X_k)$ of fuzzy real numbers is said to be convergent (uniformly) to the fuzzy real number $X_0$ if, for every $\varepsilon > 0$, there exists $m_0 \in N$ such that $\overline{d}(X_k, X_0) < \varepsilon$, for all $k \geq m_0$.

**Definition 2.9** A fuzzy real number sequence $(X_k)$ is said to be bounded if $|X_k| \leq \mu$, for some $\mu \in R^+(I)$.

**Definition 2.10** A class of sequences $E^F$ is said to be normal (or solid) if $(Y_k) \in E^F$, whenever $\overline{d}(Y_k, \overline{0}) \leq \overline{d}(X_k, \overline{0})$ for all $k \in N$ and $(X_k) \in E^F$.

**Definition 2.11** Let $K = \{k_1 < k_2 < k_3 \ldots\} \subseteq N$ and $E^F$ be a class of sequences. A $K$-step set of $E^F$ is a set of sequences $\lambda^F_k = \{(X_{k_j}) \in w^F: (X_k) \in E^F\}$.

**Definition 2.12** A canonical pre-image of a sequence $(X_{k}) \in \lambda^F_k$ is a sequence $(Y_n) \in w^F$, defined as follows:

$$Y_n = \begin{cases} X_{k_n}, & \text{if } n \in K, \\ 0, & \text{otherwise.} \end{cases}$$
Definition 2.13 A canonical pre-image of a step set \( \lambda^p_k \) is a set of canonical pre-images of all elements in \( \lambda^p_k \), i.e., \( Y \) is in canonical pre-image \( \lambda^p_k \) if and only if \( Y \) is canonical pre-image of some \( X \in \lambda^p_k \).

Definition 2.14 A class of sequences \( E^p \) is said to be monotone if \( E^p \) contains the canonical pre-images of all its step sets.

From the above definitions we have following remark.

Remark 2.15 A class of sequences \( E^p \) is solid \( \Rightarrow \) \( E^p \) is monotone.

Definition 2.16 A class of sequences \( E^p \) is said to be symmetric if \( (X_{kn}) \in E^p \), whenever \( (X_k) \in E^p \), where \( \pi \) is a permutation of \( N \).

Definition 2.17 A class of sequences \( E^p \) is said to be convergence free if \( (Y_k) \in E^p \), whenever \( (X_k) \in E^p \) and \( X_k = 0 \) implies \( Y_k = \bar{0} \).

Throughout the article \( w^p \), \( \ell^p \) and \( c^p \) denote the class of all, bounded and convergent sequences of fuzzy real numbers respectively. The class of sequences \( \ell^p \), for \( 1 \leq p < \infty \) of fuzzy real numbers was introduced and studied by Nanda (1989) as follows:

\[
\ell^p = \left\{ X = (X_k) \in w^p : \sum_{k=1}^{\infty} |d(X_k, 0)|^p < \infty \right\}.
\]

In this article we introduce the class of \( p \)-bounded variation sequences of fuzzy real numbers \( bv^p \), for \( 1 \leq p < \infty \) as follows:

\[
\ell^p = \left\{ X = (X_k) \in w^p : \sum_{k=1}^{\infty} |d(X_k, 0)|^p < \infty \right\}.
\]

where \( \Delta X_k = X_k - X_{k+1} \), for all \( k \in N \).

3. Main Results

Theorem 3.1 The class of sequences \( bv^p \), \( 1 \leq p < \infty \) is a complete metric space with the metric

\[
\rho(X, Y) = \bar{d}(X, Y) + \left[ \sum_{k=1}^{\infty} |\bar{d}(\Delta X_k, \Delta Y_k)|^p \right]^{1/p},
\]

where \( X = (X_0), Y = (Y_0) \in bv^p \).

Proof: Let \( (X^{(n)}) \) be a Cauchy sequence in \( bv^p \), where \( X^{(n)} = (X_{1}^{(n)}, X_{2}^{(n)}, X_{3}^{(n)}, \ldots) \in bv^p \), for all \( n \in N \).

Then, for each \( 0 < \varepsilon < 1 \), there exists a positive integer \( n_0 \) such that for all \( m, n \geq n_0 \),

\[
\rho(X^{(n)}, X^{(m)}) = \bar{d}(X^{(n)}, X^{(m)}) + \left[ \sum_{k=1}^{\infty} |\bar{d}(\Delta X_k^{(n)}, \Delta X_k^{(m)})|^p \right]^{1/p} < \varepsilon.
\]

It follows that \( \bar{d}(X^{(n)}, X^{(m)}) < \varepsilon \), for all \( m, n \geq n_0 \). (1)

and

\[
\left[ \sum_{k=1}^{\infty} |\bar{d}(\Delta X_k^{(n)}, \Delta X_k^{(m)})|^p \right]^{1/p} < \varepsilon,
\]

for all \( m, n \geq n_0 \). (2)

Thus \( (X^{(n)}) \) and \( (\Delta X^{(n)}) \), for all \( k \in N \) are Cauchy sequences in \( R^I \). Since \( R^I \) is complete, so \( (X^{(n)}) \) and \( (\Delta X^{(n)}) \), for all \( k \in N \) are convergent in \( R^I \).

\[
\lim_{n \to \infty} X^{(n)} = X_1
\]

and

\[
\lim_{n \to \infty} \Delta X^{(n)} = Z_k, \quad \text{for all} \quad k \in N.
\]

From (4) and (5) we have,

\[
\lim_{n \to \infty} X^{(n)} = X_k, \quad \text{for all} \quad k \in N.
\]

Now fix \( n \geq n_0 \) and let \( m \to \infty \) in (1) and (2), we have

\[
\bar{d}(X_1, X_k) < \varepsilon \quad \text{and} \quad \left[ \sum_{k=1}^{\infty} |\bar{d}(\Delta X^{(n)}, \Delta X_k)|^p \right]^{1/p} < \varepsilon,
\]

for all \( n \geq n_0 \). (6)

This implies that \( \rho(X^{(n)}, X) < \varepsilon \), for all \( n \geq n_0 \). i.e. \( X^{(n)} \to X \), as \( n \to \infty \), where \( X = (X_0) \).

Next, we show that \( X \in bv^p \).

From (6) we have for all \( n \geq n_0 \),
\[
\sum_{k=1}^{\infty} \left[ d(\Delta X_k^{(\alpha)}, \Delta X_k) \right]^p < \varepsilon.
\]

Again for all \( n \in N \), \( X_k^{(\alpha)} = (X_k^{(\alpha)}) \in bv_p^F \)

\[
\Rightarrow \sum_{k=1}^{n} \left[ d(\Delta X_k^{(\alpha)}, 0) \right]^p < \varepsilon.
\]

Now for all \( n \geq n_0 \) we have,

\[
\sum_{k=1}^{n} \left[ d(\Delta X_k^{(\alpha)}), \Delta X_k \right]^p = \sum_{k=1}^{n} \left[ d(\Delta X_k^{(\alpha)}, 0) \right]^p < \varepsilon.
\]

Hence \( X \in bv_p^F \). This proves the completeness of \( bv_p^F \).

**Theorem 3.2** The class of sequences \( bv_p^F \) is neither monotone nor solid.

**Proof:** This result follows from the following example.

**Example 3.1** Let us consider the sequence \( (X_k) \), defined as follows.

\[
X_k(t) = \begin{cases} 
2 & \text{for } 2 \leq t \leq 2 + 3^{\frac{-2}{p}}, \\
0, & \text{otherwise.}
\end{cases}
\]

Then \( [\Delta X_k] = \begin{bmatrix} 2, \frac{2}{3}(1 - \alpha)k^{\frac{2}{p}} \end{bmatrix} \) for \( k \in J \),

\[
[0,0], \quad \text{for } k \not\in J.
\]

and

\[
[\Delta Y_k] = \begin{bmatrix} 2, \frac{2}{3}(1 - \alpha)k^{\frac{2}{p}} \end{bmatrix}, \quad \text{for } k \in J,
\]

\[
\begin{bmatrix} -2, \frac{2}{3}(1 - \alpha)k^{\frac{2}{p}} \end{bmatrix}, \quad \text{for } k \not\in J.
\]

Therefore,

\[
\sum_{k=1}^{n} \left[ d(\Delta Y_k, 0) \right]^p = \sum_{k=1}^{n} \left[ 2, 3(1 - k^{2}) \right]^p.
\]

Thus \( (Y_k) \in bv_p^F \). Hence \( bv_p^F \) is not monotone.

The class \( bv_p^F \) is not solid which follows from the remark 2.15.

**Theorem 3.3** The class of sequences \( bv_p^F \) is not convergence free.

**Proof:** The result follows from the following example.

**Example 3.2** Consider the sequence \( (X_k) \in bv_p^F \) defined as follows:

For \( k \) even,

\[
X_k(t) = \begin{cases} 
1 + k^2 t, & \text{for } -k^{\frac{2}{p}} \leq t \leq 0, \\
\frac{1}{k^2} t, & \text{for } 0 < t \leq k^{\frac{2}{p}}, \\
0, & \text{otherwise}
\end{cases}
\]

and for \( k \) odd, \( X_k = 0 \).

Then

\[
[X_k] = \begin{bmatrix} (\alpha - 1)k^{\frac{2}{p}}, (1 - \alpha)k^{\frac{2}{p}} \end{bmatrix}, \quad \text{for } k \text{ even,}
\]

\[
[0,0], \quad \text{for } k \text{ odd}
\]
and
\[|\Delta X_k|^p = \begin{cases} \left( (\alpha - 1)(k+1)^{-\frac{1}{p}}, (1 - \alpha)(k+1)^{-\frac{1}{p}} \right), & \text{for } k \text{ odd}, \\ \left( (\alpha - 1)k^{-\frac{1}{p}}, (1 - \alpha)k^{-\frac{1}{p}} \right), & \text{for } k \text{ even.} \end{cases} \]
Therefore, \( \sum_{k=1}^{\infty} \left| \tilde{d}(\Delta X_k, 0) \right|^p = 2 \sum_{r=1}^{\infty} \left( 1 - \frac{\alpha}{2} \right)^p < \infty. \)
Thus \( (X_k) \in \mathcal{B}v_{p}^{F}. \)

Let us define a sequence \( (Y_k) \) as follows:
For \( k \) odd, \( Y_k = 0 \) and for \( k \) even,
\[ Y_k(t) = \begin{cases} \frac{1}{1 + \kappa^\alpha t}, & \text{for } -\kappa^{-\frac{1}{p}} \leq t \leq 0, \\ \frac{1}{1 - \kappa^\alpha t}, & \text{for } 0 < t \leq \kappa^{-\frac{1}{p}}, \end{cases} \]
\[ 0, \text{ otherwise.} \]
Then
\[ |Y_k|^p = \begin{cases} \left[ \left( (\alpha - 1)k^{-\frac{1}{p}}, (1 - \alpha)k^{-\frac{1}{p}} \right) \right], & \text{for } k \text{ even.} \end{cases} \]
and
\[ |\Delta Y_k|^p = \begin{cases} \left( (\alpha - 1)(k+1)^{-\frac{1}{p}}, (1 - \alpha)(k+1)^{-\frac{1}{p}} \right), & \text{for } k \text{ odd}, \\ \left( (\alpha - 1)k^{-\frac{1}{p}}, (1 - \alpha)(k+1)^{-\frac{1}{p}} \right), & \text{for } k \text{ even.} \end{cases} \]
Thus
\[ \sum_{k=1}^{\infty} \left| \tilde{d}(\Delta Y_k, 0) \right|^p = 2 \sum_{r=1}^{\infty} \left( 1 - \frac{\alpha}{2} \right)^p \]
which is unbounded.
\( i.e., (Y_k) \notin \mathcal{B}v_{p}^{F}. \) Hence \( \mathcal{B}v_{p}^{F}, \) is not convergence free.

**Theorem 3.4** The class of sequences \( \mathcal{B}v_{p}^{F}, \) \( p > 1 \) is not symmetric.

**Proof:** The result follows from the following example.

**Example 3.3** Consider a sequence \( (X_k) \in \mathcal{B}v_{p}^{F} \) defined as follows:

\[ X_k(t) = \begin{cases} 1, & \text{for } \frac{-1}{2} \leq t \leq 0, \\ 0, & \text{otherwise} \end{cases} \]
and for \( k \geq 2,
\[ X_k(t) = \begin{cases} 1, & \text{for } -\sum_{r=1}^{\infty} \frac{1}{r} + \frac{1}{2k} \leq t \leq -\sum_{r=1}^{\infty} \frac{1}{r}, \\ 0, & \text{otherwise} \end{cases} \]
Then \[ |X_k|^p = \begin{cases} \left[ \left( 1/k + 1/2k + 1/(k+1) \right) \right], & \text{for } k \text{ even,} \\ \left[ \left( 1/k + 1/2k + 1/(k+1) \right) \right], & \text{for } k \text{ odd.} \end{cases} \]
Now for all \( k \in \mathbb{N}, \) \[ |\Delta X_k|^p = \begin{cases} \sum_{r=1}^{\infty} \left( 1/k + 1/2k + 1/(k+1) \right)^p, & \text{for } k \text{ odd,} \\ \sum_{r=1}^{\infty} \left( 1/k + 1/2k + 1/(k+1) \right)^p, & \text{for } k \text{ even.} \end{cases} \]
For \( p > 1 \) we have,
\[ \sum_{k=1}^{\infty} \left| \tilde{d}(\Delta X_k, 0) \right|^p = \sum_{k=1}^{\infty} \left( 1/k + 1/2k + 1/(k+1) \right)^p \leq 2p \sum_{r=1}^{\infty} \left( 1/k + 1/2k + 1/(k+1) \right)^p < \infty. \]
Thus \( (X_k) \in \mathcal{B}v_{p}^{F}, \) \( p > 1. \)

Let \( (Y_k) \) be a rearrangement of the sequence \( (X_k), \) defined by
\( (Y_k) = (X_1, X_2, X_3, X_5, X_6, X_8, X_9, X_{10}, X_{11}, ... ) \).
\( i.e., Y_k = X_{\left( \frac{k+1}{2} \right)}, \) \ for \( k \) odd,
\( X_{\left( \frac{k+1}{2} \right)}, \) \ for \( k \) even and \( n \in \mathbb{N}, \) satisfying \( n(n - 1) < \frac{k}{2} \leq n(n + 1). \)

Then for \( k = 1, \) we have \[ |\Delta Y_k|^p = |X_1|^p = |X_1|^p = [0.5, 1.25]. \]
Again,
for \( k \) odd with \( k > 1 \) and \( n \in \mathbb{N}, \) satisfying \( n(n - 1) < \frac{k+1}{2} \leq n(n + 1). \)
\[
[\Delta Y_k]^p = \left[ X_k \left( \frac{1}{r(z_k)} \right)^p - X_k \left( \frac{1}{r(z_k)} - 1 \right)^p \right]^p = \\
\left[ \left\{ \left( \frac{1}{r(z_k)} \right)^p - 1 \right\} \frac{1}{2r(z_k)} \right] - \left[ \left( \frac{1}{r(z_k)} - 1 \right)^p \right] \frac{1}{2r(z_k)}
\]

and

for \( k \) even and \( n \in N \), satisfying \( n(n - 1) < \frac{k}{2} \leq n(n + 1) \).

\[
[\Delta Y_k]^p = \left[ X_k \left( \frac{1}{r(z_k)} + 1 \right)^p - X_k \left( \frac{1}{r(z_k)} - 1 \right)^p \right]^p = \\
\left[ \left\{ \left( \frac{1}{r(z_k)} \right)^p - 1 \right\} \frac{1}{2r(z_k)} \right] - \left[ \left( \frac{1}{r(z_k)} + 1 \right)^p \right] \frac{1}{2r(z_k)}
\]

It is observed that the distance of \( [\Delta Y_k]^p \) from \( [0]^p \) for all (odd and even) \( k \in N \) is numerically greater than 0.1. Therefore, \( \sum_{k=1}^{\infty} d(\Delta Y_k, 0)^p \) is unbounded for \( p > 1 \).

Thus \( (Y_k) \in BV_p^F, p > 1 \). Hence \( BV_p^F, p > 1 \) is not symmetric.

\textbf{Theorem 3.5}

\begin{enumerate}
\item \( F_p \subset \text{BV}_p^F \), for \( 1 < p < \infty \) and the inclusion is strict.
\item \( \text{BV}_q^F \subset \text{BV}_p^F \), for \( 1 \leq q < p < \infty \) and the inclusion is strict.
\item \( \text{BV}_q^F \subset \text{BV}_p^F \), for \( 1 < p < \infty \) and the inclusion is strict.
\end{enumerate}

\textbf{Proof:} (a) Let us consider a sequence \( (X_k) \in F_p \).

Then \( \sum_{k=1}^{\infty} d(\Delta X_k, 0)^p < \infty \).

Again, \( \sum_{k=1}^{\infty} d(\Delta X_k, 0)^p \leq 2^p \left\{ \sum_{k=1}^{\infty} d(\Delta X_k, 0)^p \right\} < \infty \).

Therefore, \( F_p \subset \text{BV}_p^F \).

The strictness of the inclusion follows from the following example.

\textbf{Example 3.4} Consider a sequence \( (X_k) \) defined by

\[
X_k(t) = \begin{cases} 
1 + k^p(t-2), & \text{for } 2 - k^{\frac{p}{2}} \leq t \leq 2, \\
1 - 2^{1-k}k^p(t-2), & \text{for } 2 < t \leq 2 + 2k^{\frac{p}{2}}, \\
0, & \text{otherwise}.
\end{cases}
\]

Then

\[
[\Delta X_k]^p = \left[ 2 + (\alpha - 1)k^p - 1 \right] \left\{ 2 + (1 - \alpha)k^p \right\}^p
\]

and

\[
[\Delta X_k]^p = \left[ 2(1 - \alpha)k^p + (1 - \alpha)(k + 1)^{-\frac{2}{p}} \right].
\]

Thus

\[
\sum_{k=1}^{\infty} d(\Delta X_k, 0)^p = \sum_{k=1}^{\infty} \left( 2 + (1 - \alpha)k^p \right)^p
\]

which is unbounded. Hence \( (X_k) \in F_p^F \). Next,

\[
\sum_{k=1}^{\infty} d(\Delta X_k, 0)^p \leq 2^p (1 - \alpha)^p \left\{ \sum_{k=1}^{\infty} 2^p k^p + (k + 1)^{-\frac{2}{p}} \right\} < \infty.
\]

i.e. \( (X_k) \in \text{BV}_p^F \). Therefore the inclusion is strict.

(b) Let \( (X_k) \in \text{BV}_q^F \). Then \( \sum_{k=1}^{\infty} d(\Delta X_k, 0)^p < \infty \).

Since \( \Delta X_k \to 0 \), as \( k \to \infty \), so there exists an positive integer \( n_0 \) such that

\[
d(\Delta X_k, 0) \leq 1, \text{ for all } k > n_0.
\]

We have

\[
\sum_{k=1}^{\infty} d(\Delta X_k, 0)^p = \sum_{k=1}^{n_0} d(\Delta X_k, 0)^p + \sum_{k=n_0}^{\infty} d(\Delta X_k, 0)^p. \tag{7}
\]
Clearly,
\[
\sum_{k=0}^{n} \left( \frac{1}{k+q} \right)^{p} \leq \sum_{k=0}^{n} \left( \frac{1}{k+q} \right)^{p} < \infty, \text{ for } p > q
\]
and \( \sum_{k=1}^{\infty} \left( \frac{1}{k+q} \right)^{p} \) is a finite sum.

Hence (7) \( \Rightarrow \) \( \sum_{k=1}^{\infty} \left( \frac{1}{k+q} \right)^{p} \leq 2^{p} \sum_{k=1}^{\infty} \left( \frac{1}{k+q} \right)^{p} < \infty \) \( \Rightarrow (X) \in b_{p}^{F} \)

2.5. Hence the inclusion proper.

4. Conclusions

We introduced the notion of \( p \)-bounded variation of fuzzy real number sequence, \( b_{p}^{F} \), for \( 1 \leq p < \infty \) and studied some of its properties like completeness, monotonicity, convergence free, and symmetry. We also established some inclusion results involving this space. The methodology adopted to establish the results can be applied to study the class of double bounded variation sequences. Also the space introduced can be studied from a fuzzy metric point of view.

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