Original Article

$Q$-fuzzy sets in UP-algebras

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Received: 21 March 2016; Revised: 29 July 2016; Accepted: 27 September 2016

Abstract

In this paper, we introduce the notions of $Q$-fuzzy UP-ideals and $Q$-fuzzy UP-subalgebras of UP-algebras, and their properties are investigated. Relations between a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) and a level subsets of a $Q$-fuzzy set are investigated, and conditions for a $Q$-fuzzy set to be a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) are provided. Finally, prove that it is not true that if $\mu \cdot \delta$ is a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) of $A \times B$, then either $\mu$ is a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) of $A$ or $\delta$ is a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) of $B$.

**Keywords:** UP-algebra, $Q$-fuzzy UP-ideal, $Q$-fuzzy UP-subalgebra

1. Introduction and Preliminaries

The concept of a fuzzy subset of a set was first considered by Zadeh (1965). The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere.

The concept of $Q$ fuzzy sets is introduced by many researchers and was extensively investigated in many algebraic structures such as: Jun (2001) introduced the notion of $Q$-fuzzy subalgebras of BCK/BCI-algebras. Roh et al. (2006) studied intuitionistic $Q$-fuzzy subalgebras of BCK/BCI-algebras. Muthuraj et al. (2010) introduced and investigated anti $Q$-fuzzy BG-ideals of BG-algebras. Mostafa et al. (2012) introduced the notions of $Q$-ideals and fuzzy $Q$-ideals in $Q$-algebras. Sitharselvam et al. (2012), Sithar Selvam et al. (2013) and Selvam et al. (2014) introduced and gave some properties anti $Q$-fuzzy KU-ideals, anti $Q$-fuzzy KU-subalgebras and anti $Q$-fuzzy R-closed KU-ideals of KU-algebras. The notion of anti $Q$-fuzzy R-closed PS-ideals of PS-algebras is introduced, and related properties are investigated Priya and Ramachandran (2014).

Iampan (2017) introduced a new algebraic structure, called a UP-algebra. In this paper, we introduce the notions of $Q$-fuzzy UP-ideals and $Q$-fuzzy UP-subalgebras of UP-algebras, and their properties are investigated. Relations between a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) and a level subsets of a $Q$-fuzzy set are investigated, and conditions for a $Q$-fuzzy set to be a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) are provided. Finally, prove that it is not true that if $\mu \cdot \delta$ is a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) of $A \times B$, then either $\mu$ is a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) of $A$ or $\delta$ is a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) of $B$. Before we begin our study, we will introduce the definition of a UP-algebras.

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Definition 1.1. (Jampan, 2017) An algebra $A = (A; 0)$ of type $(2, 0)$ is called a
UP-algebra if it satisfies the following axioms: for any $x, y, z \in A$,

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,

(UP-2) $0 \cdot x = x$,

(UP-3) $x \cdot 0 = 0$, and

(UP-4) $x \cdot y = y \cdot x = 0$ implies $x = y$.

In (Jampan, 2017) there is given an example of a UP-algebra.

In what follows, let $A$ and $B$ denote UP-algebras unless otherwise specified. The
following proposition is very important for the study of a UP-algebra.

Proposition 1.2. (Jampan, 2017) In a UP-algebra $A$, the following properties hold:
for any $x, y, z \in A$,

(1) $x \cdot x = 0$,

(2) $x \cdot y = 0$ and $y \cdot z = 0$ imply $x \cdot z = 0$,

(3) $x \cdot y = 0$ implies $(z \cdot x) \cdot (z \cdot y) = 0$,

(4) $x \cdot y = 0$ implies $(y \cdot z) \cdot (x \cdot z) = 0$,

(5) $x \cdot (y \cdot z) = 0$,

(6) $(y \cdot x) \cdot z = 0$ if and only if $x = y \cdot x$, and

(7) $(y \cdot y) \cdot 0$.

Definition 1.3. (Jampan, 2017) A nonempty subset $B$ of $A$ is called a UP-ideal of
$A$ if it satisfies the following properties:

(1) the constant 0 of $A$ is in $B$, and

(2) for any $x, y, z \in A, x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.

Clearly, $A$ and $\{0\}$ are UP-ideals of $A$.

Theorem 1.4. (Jampan, 2017) Let $A$ be a UP-algebra and $\{B_i\}_{i \in I}$ a family of
UP-ideals of $A$. Then $\bigcap_{i \in I} B_i$ is a UP-ideal of $A$.

Definition 1.5. (Jampan, 2017) A subset $S$ of $A$ is called a UP-subalgebra of $A$ if
the constant 0 of $A$ is in $S$, and $(S; 0)$ itself forms a UP-algebra. Clearly, $A$ and
$\{0\}$ are UP-subalgebras of $A$.

Proposition 1.6. (Jampan, 2017) A nonempty subset $S$ of a UP-algebra $A =
(A; 0)$ is a UP-subalgebra of $A$ if and only if $S$ is closed under the
multiplication on $A$.

Theorem 1.7. (Jampan, 2017) Let $A$ be a UP-algebra and $\{B_i\}_{i \in I}$ a family of
UP-subalgebras of $A$. Then $\bigcap_{i \in I} B_i$ is a UP-subalgebra of $A$.

Lemma 1.8. (Somporn et al., 2016) Let $f$ be a fuzzy set in $A$. Then the following
statements hold: for any $x, y \in A$,

(1) $\max\{f(x), f(y)\} = \min(1, f(x), 1, f(y))$, and

(2) $\min\{f(x), f(y)\} = \max(1, f(x), 1, f(y))$. 

Definition 1.9. (Kim, 2006) A Q-fuzzy set in a nonempty set \( X \) (or a Q-fuzzy subset of \( X \)) is an arbitrary function \( f: X \times X \rightarrow [0,1] \) where \( Q \) is a nonempty set and \( [0,1] \) is the unit segment of the real line.

Definition 1.10. A Q-fuzzy set \( f \) in \( A \) is called a q-fuzzy UP-ideal of \( A \) if it satisfies the following properties: for any \( x, y, z \in A \),

1. \( f(0, q) > f(x, q) \), and
2. \( f(x \cdot z, q) \geq \min\{f(x \cdot (y \cdot z), q), f(y, q)\} \).

A Q-fuzzy set \( f \) in \( A \) is called a Q-fuzzy UP-ideal of \( A \) if it is a q-fuzzy UP-ideal of \( A \) for all \( q \in Q \).

Example 1.11. Let \( A = \{0,1\} \) be a set with a binary operation \( \cdot \) defined by the following Cayley table:

\[
\begin{array}{c|cc}
\cdot & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
\end{array}
\]

Then \( (A; \cdot, 0) \) is a UP-algebra. Let \( Q = \{a, b\} \). We define a Q-fuzzy set \( f \) in \( A \) as follows:

\[
\begin{array}{c|cc}
f & a & b \\
0 & 0.3 & 0.2 \\
1 & 0.1 & 0.1 \\
\end{array}
\]

Using this data, we can show that \( f \) is a Q-fuzzy UP-ideal of \( A \).

Example 1.12. Let \( A = \{0,1\} \) be a set with a binary operation \( \cdot \) defined by the following Cayley table:

\[
\begin{array}{c|cc}
\cdot & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
\end{array}
\]

Then \( (A; \cdot, 0) \) is a UP-algebra. Let \( Q = \{a, b\} \). We define a Q-fuzzy set \( f \) in \( A \) as follows:

\[
\begin{array}{c|cc}
f & a & b \\
0 & 0.3 & 0.1 \\
1 & 0.1 & 0.2 \\
\end{array}
\]

By Example 1.11, we have \( f \) is an \( a \)-fuzzy UP-ideal of \( A \). Since \( f(0, b) = 0.1 < 0.2 = f(1, b) \), we have Definition 1.10 (1) is false. Therefore, \( f \) is not a \( b \)-fuzzy UP-ideal of \( A \). Hence, \( f \) is not a Q-fuzzy UP-ideal of \( A \).

Definition 1.13. A Q-fuzzy set \( f \) in \( A \) is called a q-fuzzy UP-subalgebra of \( A \) if for any \( x, y \in A \),

\[
f(x \cdot y, q) \geq \min\{f(x, q), f(y, q)\}.
\]

A Q-fuzzy set \( f \) in \( A \) is called a Q-fuzzy UP-subalgebra of \( A \) if it is a q-fuzzy UP-subalgebra of \( A \) for all \( q \in Q \).

Example 1.14. Let \( A = \{0,1,2\} \) be a set with a binary operation \( \cdot \) defined by the following Cayley table:

\[
\begin{array}{c|ccc}
\cdot & 0 & 1 & 2 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
2 & 0 & 0 & 0 \\
\end{array}
\]

Then \( (A; \cdot, 0) \) is a UP-algebra. Let \( Q = \{a, b\} \). We defined a Q-fuzzy set \( f \) in \( A \) as follows:
Using this data, we can show that \( f \) is a \( Q \)-fuzzy UP-subalgebra of \( A \).

**Example 1.15.** Let \( A = \{0, 1, 2\} \) be a set with a binary operation \( \cdot \) defined by the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \( (A; \cdot, 0) \) is a UP-algebra. Let \( Q = \{a, b\} \). We defined a \( Q \)-fuzzy set \( f \) in \( A \) as follows:

<table>
<thead>
<tr>
<th></th>
<th>( a )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.4</td>
<td>0.7</td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td>2</td>
<td>0.3</td>
<td>0.5</td>
</tr>
</tbody>
</table>

By Example 1.14, we have \( f \) is an \( \alpha \)-fuzzy UP-subalgebra of \( A \). Since \( f(1 \cdot 1, b) = 0.1 < 0.5 = \min\{f(1, b), f(1, b)\} \), we have Definition 1.13 is false. Therefore, \( f \) is not a \( b \)-fuzzy UP-subalgebra of \( A \). Hence, \( f \) is not a \( Q \)-fuzzy UP-subalgebra of \( A \).

**Definition 1.16.** (Kim, 2006) Let \( f \) be a \( Q \)-fuzzy set in \( A \). The \( Q \)-fuzzy set \( \bar{f} \) defined by \( \bar{f}(x, q) = 1 - f(x, q) \) for all \( x \in A \) and \( q \in Q \) is called the complement of \( f \) in \( A \).

**Remark 1.17.** For all \( Q \)-fuzzy set \( f \) in \( A \), we have \( f = \overline{\overline{f}} \).

**Definition 1.18.** Let \( f \) be a \( Q \)-fuzzy set in \( A \). For any \( t \in [0, 1] \), the sets

\[
U(f; t) = \{ x \in A \mid f(x, q) \geq t \text{ for all } q \in Q \}
\]

and

\[
U^+(f; t) = \{ x \in A \mid f(x, q) > t \text{ for all } q \in Q \}
\]

are called an upper \( t \)-level subset and an upper \( t \)-strong level subset of \( f \), respectively. The sets

\[
L(f; t) = \{ x \in A \mid f(x, q) \leq t \text{ for all } q \in Q \}
\]

and

\[
L^-(f; t) = \{ x \in A \mid f(x, q) < t \text{ for all } q \in Q \}
\]

are called a lower \( t \)-level subset and a lower \( t \)-strong level subset of \( f \), respectively. For any \( q \in Q \), the sets

\[
U(f; t, q) = \{ x \in A \mid f(x, q) \geq t \}
\]

and

\[
U^+(f; t, q) = \{ x \in A \mid f(x, q) > t \}
\]

are called a \( q \)-upper \( t \)-level subset and a \( q \)-upper \( t \)-strong level subset of \( f \), respectively. The sets

\[
L(f; t, q) = \{ x \in A \mid f(x, q) \leq t \}
\]
and
\[ L^-(f; t, q) = \{ x \in A \mid f(x, q) < t \} \]
are called a \textit{q-lower t-level subset} and a \textit{q-lower t-strong level subset} of \( f \), respectively.

We can easily prove the following two remarks.

\textbf{Remark 1.19.} Let \( f \) be a \( Q \)-fuzzy set in \( A \) and for any \( t_1, t_2 \in [0, 1] \) with \( t_1 \leq t_2 \).
Then the following properties hold:
1. \( L(f; t_1) \subseteq L(f; t_2) \),
2. \( U(f; t_2) \subseteq U(f; t_1) \),
3. \( L^-(f; t_1) \subseteq L^-(f; t_2) \), and
4. \( U^+(f; t_2) \subseteq U^+(f; t_1) \).

\textbf{Remark 1.20.} Let \( f \) be a \( Q \)-fuzzy set in \( A \) and for any \( t_1, t_2 \in [0, 1] \) with \( t_1 \leq t_2 \) and \( q \in Q \).
Then the following properties hold:
1. \( L(f; t_1, q) \subseteq L(f; t_2, q) \),
2. \( U(f; t_2, q) \subseteq U(f; t_1, q) \),
3. \( L^-(f; t_1, q) \subseteq L^-(f; t_2, q) \), and
4. \( U^+(f; t_2, q) \subseteq U^+(f; t_1, q) \).

\textbf{Definition 1.21.} (Tampan, 2017) Let \( (A; \cdot, 0) \) and \( (A'; \cdot', 0') \) be \( UP \)-algebras. A mapping \( f \) from \( A \) to \( A' \) is called a \textit{UP-homomorphism} if
\[ f(x \cdot y) = f(x) \cdot' f(y) \]
for all \( x, y \in A \).

A \textit{UP-homomorphism} \( f; A \rightarrow A' \) is called a
1. \textit{UP-endomorphism} of \( A \) if \( A' = A \),
2. \textit{UP-epimorphism} if \( f \) is surjective,
3. \textit{UP monomorphism} if \( f \) is injective, and
4. \textit{UP-isomorphism} if \( f \) is bijective. Moreover, we say \( A \) is \textit{UP-isomorphic to} \( A' \), symbolically \( A \cong A' \), if there is a \textit{UP-isomorphism} from \( A \) to \( A' \).

\textbf{Proposition 1.22.} (Tampan, 2017) Let \( (A; \cdot, 0_A) \) and \( (B; \cdot, 0_B) \) be \( UP \)-algebras and let \( f; A \rightarrow B \) be a \( UP \)-homomorphism. Then \( f(0_A) = 0_B \).

\textbf{Definition 1.23.} (Sithar Selvam et al., 2013) Let \( f; A \rightarrow B \) be a function and \( \mu \) be a \( Q \)-fuzzy set in \( B \). We define a new \( Q \)-fuzzy set in \( A \) by \( \mu_f \) as
\[ \mu_f(x, q) = \mu(f(x), q) \]
for all \( x \in A \) and \( q \in Q \).

\textbf{Definition 1.24.} (Sithar Selvam et al., 2013) Let \( f; A \rightarrow B \) be a bijection and \( \mu_f \) be a \( Q \)-fuzzy set in \( A \). We define a new \( Q \)-fuzzy set in \( B \) by \( \mu \) as
\[ \mu(y, q) = \mu_f(x, q) \]
where \( f(x) = y \) for all \( x \in B \) and \( q \in Q \).

\textbf{Definition 1.25.} (Sithar Selvam et al., 2013) Let \( \mu \) be a \( Q \)-fuzzy set in \( A \) and \( \delta \) be a \( Q \)-fuzzy set in \( B \). The \textit{Cartesian product} \( \mu \times \delta; (A \times B) \times Q \rightarrow [0, 1] \) is defined by
\[ (\mu \times \delta)((x, y), q) = \max\{\mu(x, q), \delta(y, q)\} \]
for all \( x \in A, y \in B \) and \( q \in Q \).
The dot product $\mu \cdot \delta : (A \times B) \times Q \rightarrow [0, 1]$ is defined by
\[
(\mu \cdot \delta)((x, y), q) = \min\{\mu(x, q), \delta(y, q)\} \text{ for all } x \in A, y \in B \text{ and } q \in Q.
\]

2 Main Results

In this section, we study $Q$-fuzzy UP-ideals and $Q$-fuzzy UP-subalgebras of UP-algebras, and their properties are investigated. Relations between a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) and a level subsets of a $Q$-fuzzy set are investigated, and conditions for a $Q$-fuzzy set to be a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) are provided. Finally, prove that it is not true that if $\mu \cdot \delta$ is a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) of $A \times B$, then either $\mu$ is a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) of $A$ or $\delta$ is a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) of $B$.

Theorem 2.1. Every $q$-fuzzy UP-ideal of $A$ is a $q$-fuzzy UP-subalgebra of $A$.

Proof. Let $f$ be a $q$-fuzzy UP-ideal of $A$. Let $x, y \in A$. Then
\[
f(x \cdot y, q) \geq \min \{f(x \cdot (y, q), q), f(y, q)\} \quad \text{(Definition 1.10 (2))}
\]
\[
= \min \{f(x \cdot 0, q), f(y, q)\} \quad \text{(Proposition 1.2 (1))}
\]
\[
= \min \{f(0, q), f(y, q)\} \quad \text{(UP 3)}
\]
\[
= f(y, q) \quad \text{(Definition 1.10 (1))}
\]
\[
> \min \{f(x, q), f(y, q)\}.
\]

Hence, $f$ is a $q$-fuzzy UP-subalgebra of $A$.

With Definition 1.10 and Theorem 2.1, we obtain the corollary.

Corollary 2.2. Every $Q$-fuzzy UP-ideal of $A$ is a $Q$-fuzzy UP-subalgebra of $A$.

Theorem 2.3. If $f$ is a $q$-fuzzy UP-subalgebra of $A$, then $f(0, q) > f(x, q)$ for all $x \in A$.

Proof. Assume that $f$ is a $q$-fuzzy UP-subalgebra of $A$. By Proposition 1.2 (1), we have $f(0, q) = f(x \cdot x, q) \geq \min \{f(x, q), f(x, q)\} = f(x, q)$ for all $x \in A$.

With Definition 1.13 and Theorem 2.3, we obtain the corollary.

Corollary 2.4. If $f$ is a $Q$-fuzzy UP-subalgebra of $A$, then $f(0, q) \geq f(x, q)$ for all $x \in A$ and $q \in Q$.

We can easily prove the following three lemmas.

Lemma 2.5. Let $f$ be a $Q$-fuzzy set in $A$ and for any $t \in [0, 1]$. Then the following properties hold:

1. $L(f; t) = U(t; 1 - t)$,
2. $L(f; t) = U(1 - t; 1 - t)$,
3. $U(f; t) = L(t; 1 - t)$, and
4. $U^+(f; t) = L^-(f; 1 - t)$. 

Lemma 2.6. Let $f$ be a $Q$-fuzzy set in $A$ and for any $t \in [0,1]$ and $q \in Q$. Then the following properties hold:

(i) $L(f; t, q) = U(\overline{f}; 1 - t, q)$,

(ii) $L^-(f; t, q) = U^+(\overline{f}; 1 - t, q)$,

(iii) $U(f; t, q) = L(\overline{f}; 1 - t, q)$, and

(iv) $U^-(f; t, q) = L^+(\overline{f}; 1 - t, q)$.

Lemma 2.7. Let $f$ be a $Q$-fuzzy set in $A$ and for any $t \in [0,1]$ and $q \in Q$. Then the following properties hold:

(i) $L(f; t) \subseteq \bigcap_{q \in Q} L(f; t, q)$,

(ii) $L^-(f; t) \subseteq \bigcap_{q \in Q} L^-(f; t, q)$,

(iii) $U(f; t) \subseteq \bigcap_{q \in Q} U(f; t, q)$, and

(iv) $U^-(f; t) \subseteq \bigcap_{q \in Q} U^-(f; t, q)$.

Lemma 2.8. (Minik and Ansari, 2018) For any $a, b \in K$ such that $a < b$, $a < \frac{1}{2} a < b$.

Theorem 2.9. Let $f$ be a $Q$-fuzzy set in $A$. Then the following statements hold:

(i) $\overline{f}$ is a $Q$-fuzzy $UP$-ideal of $A$ if and only if the following condition (a) holds: for any $t \in [0,1]$ and $q \in Q$, $L(f; t, q)$ is either empty or a $UP$-ideal of $A$,

(ii) $\overline{f}$ is a $Q$-fuzzy $UP$-ideal of $A$ if and only if the following condition (a) holds: for any $t \in [0,1]$ and $q \in Q$, $L^-(f; t, q)$ is either empty or a $UP$-ideal of $A$,

(iii) $f$ is a $Q$-fuzzy $UP$-ideal of $A$ if and only if the following condition (a) holds: for any $t \in [0,1]$ and $q \in Q$, $U(f; t, q)$ is either empty or a $UP$-ideal of $A$, and

(iv) $f$ is a $Q$-fuzzy $UP$-ideal of $A$ if and only if the following condition (a) holds: for any $t \in [0,1]$ and $q \in Q$, $U^-(f; t, q)$ is either empty or a $UP$-ideal of $A$.

Proof. (1) Assume that $\overline{f}$ is a $Q$-fuzzy $UP$-ideal of $A$. Then $\overline{f}$ is a $q$-fuzzy $UP$-ideal of $A$ for all $q \in Q$. Let $q \in Q$ and $t \in [0,1]$ be such that $\overline{f}(f; t, q) \neq \emptyset$ and let $x \in L(f; t, q)$. Then $f(x, q) < t$. Now,

\[
\overline{f}(0, q) - f(x, 0, q) \geq \min(\overline{f}(x \cdot (x, 0, q)), \overline{f}(x, q)) \quad \text{(UP-3)}
\]

\[
\min(\overline{f}(x \cdot 0, q), \overline{f}(x, q)) \quad \text{(UP-3)}
\]

\[
\min(\overline{f}(0, q), \overline{f}(x, q)) \quad \text{(UP-3)}
\]

\[
f(x, q). \quad \text{(Definition 1.10 (1))}
\]

Then $1 - f(0, q) > 1 - f(x, q)$, so $f(0, q) < f(x, q) < t$. Hence, $0 \in L(f; t, q)$. Let $x, y, z \in A$ be such that $x \cdot (y, z) \in L(f; t, q)$ and $y \in L(f; t, q)$. Then $f(x \cdot (y, z), q) \leq t$ and $f(y, q) \leq t$. By Definition 1.10 (2), we have $\overline{f}(x \cdot (y, z), q) \geq \min(\overline{f}(x \cdot (y, z), q), f(y, q))$. Thus

\[
f(x \cdot (y, z), q) \geq \min\{1, f(x \cdot (y, z), q), f(y, q)\}
\]

\[
eq 1 - \max\{f(x \cdot (y, z), q), f(y, q)\} \quad \text{(Lemma 1.8 (1))}
\]
Then \( f(x \cdot z, q) \leq \max \{ f(x \cdot (y \cdot z), q), f(y, q) \} \leq t \). Hence, \( x \cdot z \in L(f; t, q) \). Therefore, \( L(f; t, q) \) is a UP-ideal of \( A \).

Conversely, assume that the condition (\( \ast \)) holds and suppose that \( \overline{f}(0, q) > \overline{f}(x, q) \) for all \( x \in A \) and \( q \in Q \) is false. Then there exist \( x \in A \) and \( q \in Q \) such that \( \overline{f}(0, q) < \overline{f}(x, q) \). Thus \( 1 - f(0, q) < 1 - f(x, q) \), so \( f(0, q) > f(x, q) \). Let \( t = \frac{f(0, q) + f(x, q)}{2} \). Then \( t \in [0, 1] \) and by Lemma 2.8, we have \( f(0, q) > t > f(x, q) \). Thus \( x \in L(f; t, q) \), so \( L(f; t, q) \) \( \neq \emptyset \). By assumption, we have \( L(f; t, q) \) is a UP-ideal of \( A \). It follows that \( 0 \in L(f; t, q) \), so \( f(0, q) \leq t \) which is a contradiction. Hence, \( \overline{f}(0, q) \geq \overline{f}(x, q) \) for all \( x \in A \) and \( q \in Q \). Suppose that \( \overline{f}(x \cdot z, q) \geq \min \{ \overline{f}(x \cdot (y \cdot z), q), \overline{f}(y, q) \} \) for all \( x, y, z \in A \) and \( q \in Q \) is false. Then there exist \( x, y, z \in A \) and \( q \in Q \) such that \( \overline{f}(x \cdot z, q) < \min \{ \overline{f}(x \cdot (y \cdot z), q), \overline{f}(y, q) \} \). Thus

\[
1 - f(x \cdot z, q) < \min \{ 1 - f(x \cdot (y \cdot z), q), 1 - f(y, q) \} = 1 - \max \{ f(x \cdot (y \cdot z), q), f(y, q) \}.
\]

Then \( f(x \cdot z, q) > \max \{ f(x \cdot (y \cdot z), q), f(y, q) \} \). Let \( g_0 = \frac{f(x \cdot z, q) + \max \{ f(x \cdot (y \cdot z), q), f(y, q) \}}{2} \). Then \( g_0 \in [0, 1] \) and by Lemma 2.8, we have \( f(x \cdot z, q) > g_0 \) \( \geq \max \{ f(x \cdot (y \cdot z), q), f(y, q) \} \). Thus \( f(x \cdot (y \cdot z), q) < g_0 \) and \( f(y, q) < g_0 \), so \( x \cdot (y \cdot z) \in L(f; g_0, q) \) and \( y \in L(f; g_0, q) \), so \( L(f; g_0, q) \) \( \neq \emptyset \). By assumption, we have \( L(f; g_0, q) \) is a UP-ideal of \( A \). It follows that \( x \cdot z \in L(f; g_0, q) \), so \( f(x \cdot z, q) \leq g_0 \) which is a contradiction. Hence, \( \overline{f}(x \cdot z, q) \geq \min \{ \overline{f}(x \cdot (y \cdot z), q), \overline{f}(y, q) \} \) for all \( x, y, z \in A \) and \( q \in Q \). Consequently, \( \overline{f} \) is a UP-fuzzy UP-ideal of \( A \).

(2) Similarly to as in the proof of (1).

(3) Assume that \( f \) is a UP-fuzzy UP-ideal of \( A \). Then \( f \) is a UP-fuzzy UP-ideal of \( A \) for all \( q \in Q \). Let \( q \in Q \) and \( t \in [0, 1] \) be such that \( U(f; t, q) \neq \emptyset \) and let \( x \in U(f; t, q) \). Then \( f(x, q) \geq t \). Now,

\[
f(0, q) = f(x \cdot 0, q)
\geq \min \{ f(x \cdot (x \cdot 0), q), f(x, q) \} \quad \text{(Definition 1.10 (2))}
= \min \{ f(x \cdot 0, q), f(x, q) \} \quad \text{(UP-3)}
= \min \{ f(0, q), f(x, q) \} \quad \text{(UP-3)}
= f(x, q) \quad \text{(Definition 1.10 (1))}
\geq t.
\]

Hence, \( 0 \in U(f; t, q) \). Let \( x, y, z \in A \) be such that \( x \cdot (y \cdot z) \in U(f; t, q) \) and \( y \in U(f; t, q) \). Then \( f(x \cdot (y \cdot z), q) \geq t \) and \( f(y, q) \geq t \). By Definition 1.10 (2), we have \( f(x \cdot z, q) \geq \min \{ f(x \cdot (y \cdot z), q), f(y, q) \} \geq t \). Thus \( x \cdot z \in U(f; t, q) \). Hence, \( U(f; t, q) \) is a UP-ideal of \( A \).

Conversely, assume that the condition (\( \ast \)) holds and suppose that \( f(0, q) \geq f(x, q) \) for all \( x \in A \) and \( q \in Q \) is false. Then there exist \( x \in A \) and \( q \in Q \) such that \( f(0, q) < f(x, q) \). Let \( t = \frac{f(0, q) + f(x, q)}{2} \). Then \( t \in [0, 1] \) and by Lemma 2.8, we have \( f(0, q) < t < f(x, q) \). Thus \( x \in U(f; t, q) \), so \( U(f; t, q) \neq \emptyset \). By assumption, we have \( U(f; t, q) \) is a UP-ideal of \( A \). It follows that \( 0 \in U(f; t, q) \), so \( f(0, q) \geq t \) which is a contradiction. Hence, \( f(0, q) \geq f(x, q) \) for all \( x \in A \) and \( q \in Q \). Suppose that \( f(x \cdot z, q) \geq \min \{ f(x \cdot (y \cdot z), q), f(y, q) \} \) for all \( x, y, z \in A \) and \( q \in Q \) is false. Then there exist \( x, y, z \in A \) and \( q \in Q \) such that \( f(x \cdot z, q) < \min \{ f(x \cdot (y \cdot z), q), f(y, q) \} \). Let \( g_0 = \frac{f(x \cdot z, q) + \min \{ f(x \cdot (y \cdot z), q), f(y, q) \}}{2} \). Then \( g_0 \in [0, 1] \) and by Lemma 2.8, we have \( f(x \cdot z, q) < g_0 < \min \{ f(x \cdot (y \cdot z), q), f(y, q) \} \). Thus
\[ f(x \cdot (y \cdot z), q) > g_0 \text{ and } f(y, q) > g_0, \text{ so } x \cdot (y \cdot z) \in U(f; g_0, q) \text{ and } y \in U(f; g_0, q), \]

so \( U(f; g_0, q) \neq \emptyset \). By assumption, we have \( U(f; g_0, q) \) is a UP-ideal of \( A \). It follows that \( x \cdot z \in U(f; g_0, q) \), so \( f(x \cdot z, q) \geq g_0 \) which is a contradiction. Hence, \( f(\tau \cdot \zeta, q) = \min \{ f(\tau \cdot (y \cdot z), q), f(y, q) \} \) for all \( \tau, \zeta, y, z \in A \) and \( q \in Q \). Therefore, \( f \) is a \( Q \)-fuzzy UP-ideal of \( A \) for all \( q \in Q \). Consequently, \( f \) is a \( Q \)-fuzzy UP-ideal of \( A \).

(4) Similarly to as in the proof of (3).

Corollary 2.10. Let \( f \) be a \( Q \)-fuzzy set in \( A \). Then the following statements hold:

(1) if \( f \) is a \( Q \)-fuzzy UP-ideal of \( A \), then for any \( t \in [0, 1] \), \( L(f; t) \) is either empty or a UP-ideal of \( A \),

(2) if \( f \) is a \( Q \)-fuzzy UP-ideal of \( A \), then for any \( t \in [0, 1] \), \( L(f; t) \) is either empty or a UP-ideal of \( A \),

(3) if \( f \) is a \( Q \)-fuzzy UP-ideal of \( A \), then for any \( t \in [0, 1] \), \( U(f; t) \) is either empty or a UP-ideal of \( A \), and

(4) if \( f \) is a \( Q \)-fuzzy UP-ideal of \( A \), then for any \( t \in [0, 1] \), \( U^+(f; t) \) is either empty or a UP-ideal of \( A \).

Proof. (1) Assume that \( f \) is a \( Q \)-fuzzy UP-ideal of \( A \). By Theorem 2.9 (1), we have that for any \( t \in [0, 1] \) and \( q \in Q \), \( L(f; t, q) \) is either empty or a UP-ideal of \( A \). Let \( t \in [0, 1] \). If \( L(f; t, q) \neq \emptyset \) for some \( q \in Q \), it follows from Lemma 2.7 (1) that \( L(f; t, q) = \bigcap_{q \in Q} L(f; t, q) = \emptyset \). If \( L(f; t, q) = \emptyset \) for all \( q \in Q \), it follows from Theorem 2.9 (1) that \( L(f; t, q) \) is a UP-ideal of \( A \) for all \( q \in Q \). By Lemma 2.7 (1) and Theorem 1.4, we have \( L(f; t) = \bigcap_{q \in Q} L(f; t, q) \) is a UP-ideal of \( A \).

(2) Similarly to as in the proof of (1).

(3) Assume that \( f \) is a \( Q \)-fuzzy UP-ideal of \( A \). By Theorem 2.9 (3), we have that for any \( t \in [0, 1] \) and \( q \in Q \), \( U(f; t, q) \) is either empty or a UP-ideal of \( A \). Let \( t \in [0, 1] \). If \( U(f; t, q) \neq \emptyset \) for some \( q \in Q \), it follows from Lemma 2.7 (3) that \( U(f; t, q) = \bigcap_{q \in Q} U(f; t, q) = \emptyset \). If \( U(f; t, q) \neq \emptyset \) for all \( q \in Q \), it follows from Theorem 2.9 (3) that \( U(f; t, q) \) is a UP ideal of \( A \) for all \( q \in Q \). By Lemma 2.7 (3) and Theorem 1.4, we have \( U(f; t) = \bigcap_{q \in Q} U(f; t, q) \) is a UP ideal of \( A \).

(4) Similarly to as in the proof of (3).

Theorem 2.11. Let \( f \) be a \( Q \)-fuzzy set in \( A \). Then the following statements hold:

(1) \( f \) is a \( Q \)-fuzzy UP-subalgebra of \( A \) if and only if the following condition (s) holds, for any \( t \in [0, 1] \) and \( q \in Q \), \( L(f; t, q) \) is either empty or a UP-subalgebra of \( A \),

(2) \( f \) is a \( Q \)-fuzzy UP-subalgebra of \( A \) if and only if the following condition (s) holds, for any \( t \in [0, 1] \) and \( q \in Q \), \( L^-\( f; t, q \) \) is either empty or a UP-subalgebra of \( A \),

(3) \( f \) is a \( Q \)-fuzzy UP-subalgebra of \( A \) if and only if the following condition (s) holds, for any \( t \in [0, 1] \) and \( q \in Q \), \( U^-(f; t, q) \) is either empty or a UP-subalgebra of \( A \), and

(4) \( f \) is a \( Q \)-fuzzy UP-subalgebra of \( A \) if and only if the following condition (s) holds, for any \( t \in [0, 1] \) and \( q \in Q \), \( U^+(f; t, q) \) is either empty or a UP-subalgebra of \( A \).
Proof. (1) Assume that \( \overline{f} \) is a \( q \)-fuzzy UP-subalgebra of \( A \). Then \( \overline{f} \) is a \( q \)-fuzzy UP-subalgebra of \( A \) for all \( q \in Q \). Let \( q \in Q \) and \( t \in [0,1] \) be such that \( L(f; t, q) \neq \emptyset \) and let \( x, y \in L(f; t, q) \). Then \( f(x, q) \leq t \) and \( f(y, q) \leq t \). Now,

\[
\overline{f}(x \cdot y, q) = \min \{ \overline{f}(x, q), \overline{f}(y, q) \} \\
= \min \{ 1 - f(x, q), 1 - f(y, q) \} \\
= 1 - \max \{ f(x, q), f(y, q) \}.
\]

(1.8) \( \Box \)

Then \( f(x \cdot y, q) \leq \max \{ f(x, q), f(y, q) \} \leq t \), so \( x \cdot y \in L(f; t, q) \). Hence, \( L(f; t, q) \) is a UP-subalgebra of \( A \).

Conversely, assume that the condition (\( \ast \)) holds. Let \( x, y \in A \) and \( q \in Q \) and let \( t = \max \{ f(x, q), f(y, q) \} \). Thus \( f(x, q) \leq t \) and \( f(y, q) \leq t \), so \( x, y \in L(f; t, q) \neq \emptyset \).

By assumption, we have \( L(f; t, q) \) is a UP-subalgebra of \( A \). It follows that \( x \cdot y \in L(f; t, q) \). Thus \( f(x \cdot y, q) \leq t = \max \{ f(x, q), f(y, q) \} \), so

\[
1 - f(x \cdot y, q) \geq 1 - \max \{ f(x, q), f(y, q) \} \\
= \min \{ 1 - f(x, q), 1 - f(y, q) \}.
\]

(1.8) \( \Box \)

Hence, \( f(x \cdot y, q) \geq \min \{ f(x, q), f(y, q) \} \). Therefore, \( f \) is a \( q \)-fuzzy UP-subalgebra of \( A \) for all \( q \in Q \). Consequently, \( \overline{f} \) is a \( q \)-fuzzy UP-subalgebra of \( A \).

(2) Similarly to as in the proof of the necessity of (1).

Conversely, assume that the condition (\( \ast \)) holds. Assume that there exist \( x, y \in A \) and \( q \in Q \) such that \( \overline{f}(x \cdot y, q) < \min \{ \overline{f}(x, q), \overline{f}(y, q) \} \). By Lemma 1.8 (1), we have

\[
1 - f(x \cdot y, q) < \min \{ 1 - f(x, q), 1 - f(y, q) \} = 1 - \max \{ f(x, q), f(y, q) \}.
\]

Thus \( f(x \cdot y, q) > \max \{ f(x, q), f(y, q) \} \). Now \( f(x \cdot y, q) \in [0,1] \), we choose \( t = f(x \cdot y, q) \). Thus \( f(x, q) \leq t \) and \( f(y, q) \leq t \), so \( x, y \in L(f; t, q) \neq \emptyset \).

By assumption, we have \( L(f; t, q) \) is a UP-subalgebra of \( A \) and so \( x \cdot y \in L(f; t, q) \). Thus \( f(x \cdot y, q) < t = f(x, y, q) \) which is a contradiction. Hence, \( \overline{f}(x \cdot y, q) \geq \min \{ \overline{f}(x, q), \overline{f}(y, q) \} \) for all \( x, y \in A \) and \( q \in Q \). Therefore, \( \overline{f} \) is a \( q \)-fuzzy UP-subalgebra of \( A \) for all \( q \in Q \). Consequently, \( \overline{f} \) is a \( q \)-fuzzy UP-subalgebra of \( A \).

(3) Assume that \( f \) is a \( q \)-fuzzy UP-subalgebra of \( A \). Then \( f \) is a \( q \)-fuzzy UP-subalgebra of \( A \) for all \( q \in Q \). Let \( q \in Q \) and \( t \in [0,1] \) be such that \( U(f; t, q) \neq \emptyset \) and let \( x, y \in U(f; t, q) \). Then \( f(x, q) \geq t \) and \( f(y, q) \geq t \), we have \( f(x \cdot y, q) \geq \min \{ f(x, q), f(y, q) \} \geq t \). Thus \( x \cdot y \in U(f; t, q) \). Hence, \( U(f; t, q) \) is a UP-subalgebra of \( A \).

Conversely, assume that the condition (\( \ast \)) holds. Let \( x, y \in A \) and \( q \in Q \) and let \( t = \min \{ f(x, q), f(y, q) \} \). Thus \( f(x, q) \geq t \) and \( f(y, q) \geq t \), so \( x, y \in U(f; t, q) \neq \emptyset \).

By assumption, we have \( U(f; t, q) \) is a UP-subalgebra of \( A \). It follows that \( x \cdot y \in U(f; t, q) \). Thus \( f(x \cdot y, q) \geq t = \min \{ f(x, q), f(y, q) \} \). Hence, \( f \) is a \( q \)-fuzzy UP-subalgebra of \( A \) for all \( q \in Q \). Consequently, \( f \) is a \( q \)-fuzzy UP-subalgebra of \( A \).

(4) Similarly to as in the proof of the necessity of (3).

Conversely, assume that the condition (\( \ast \)) holds. Assume that there exist \( x, y \in A \) and \( q \in Q \) such that \( f(x \cdot y, q) < \min \{ f(x, q), f(y, q) \} \). Then \( f(x \cdot y, q) \in [0,1] \). Choose \( t = f(x \cdot y, q) \). Thus \( f(x, q) > t \) and \( f(y, q) > t \), so \( x, y \in U^1(f; t, q) \neq \emptyset \).

By assumption, we have \( U^1(f; t, q) \) is a UP-subalgebra of \( A \) and so \( x \cdot y \in U^1(f; t, q) \). Thus \( f(x \cdot y, q) > t = f(x, y, q) \) which is a contradiction. Hence, \( f(x \cdot y, q) \geq \min \{ f(x, q), f(y, q) \} \) for all \( x, y \in A \) and \( q \in Q \). Therefore, \( f \) is a \( q \)-fuzzy UP-subalgebra of \( A \).
Corollary 2.12. Let $f$ be a $Q$-fuzzy set in $A$. Then the following statements hold:

1. if $\overline{f}$ is a $Q$-fuzzy UP-subalgebra of $A$, then for any $t \in [0,1]$, $L(f;t)$ is either empty or a UP-subalgebra of $A$,

2. if $\overline{f}$ is a $Q$-fuzzy UP-subalgebra of $A$, then for any $t \in [0,1]$, $L^-(f;t)$ is either empty or a UP-subalgebra of $A$,

3. if $f$ is a $Q$-fuzzy UP-subalgebra of $A$, then for any $t \in [0,1]$, $U(f;t)$ is either empty or a UP-subalgebra of $A$, and

4. if $f$ is a $Q$-fuzzy UP-subalgebra of $A$, then for any $t \in [0,1]$, $U^+(f;t)$ is either empty or a UP-subalgebra of $A$.

Proof. (1) Assume that $\overline{f}$ is a $Q$-fuzzy UP-subalgebra of $A$. By Theorem 2.11 (1), we have for any $t \in [0,1]$ and $q \in Q$, $L(f;t,q)$ is either empty or a UP-subalgebra of $A$. Let $t \in [0,1]$. If $L(f;t,q) = \emptyset$ for some $q \in Q$, it follows from Lemma 2.7 (1) that $L(f;t) = \bigcap_{q \in Q} L(f;t,q) = \emptyset$. If $L(f;t,q) \neq \emptyset$ for all $q \in Q$, it follows from Theorem 2.11 (1) that $L(f;t,q)$ is a UP-subalgebra of $A$ for all $q \in Q$. By Lemma 2.7 (1) and Theorem 1.7, we have $L(f;t) = \bigcap_{q \in Q} L(f;t,q)$ is a UP-subalgebra of $A$.

(2) Similarly to as in the proof of (1).

(3) Assume that $f$ is a $Q$-fuzzy UP-subalgebra of $A$. By Theorem 2.11 (3), we have for any $t \in [0,1]$ and $q \in Q$, $U(f;t,q)$ is either empty or a UP-subalgebra of $A$. Let $t \in [0,1]$. If $U(f;t,q) = \emptyset$ for some $q \in Q$, it follows from Lemma 2.7 (3) that $U(f;t) = \bigcap_{q \in Q} U(f;t,q) = \emptyset$. If $U(f;t,q) \neq \emptyset$ for all $q \in Q$, it follows from Theorem 2.11 (3) that $U(f;t,q)$ is a UP-subalgebra of $A$ for all $q \in Q$. By Lemma 2.7 (3) and Theorem 1.7, we have $U(f;t) = \bigcap_{q \in Q} U(f;t,q)$ is a UP-subalgebra of $A$.

(4) Similarly to as in the proof of (3).

Corollary 2.13. Let $I$ be a UP-ideal of $A$. Then the following statements hold:

1. for any $k \in (0,1]$, then there exists a $Q$-fuzzy UP-ideal $g$ of $A$ such that $L(g;t) = I$ for all $t < k$ and $L(g;t) = A$ for all $t \geq k$, and

2. for any $k \in [0,1)$, then there exists a $Q$-fuzzy UP-ideal $f$ of $A$ such that $U(f;t) = I$ for all $t > k$ and $U(f;t) = A$ for all $t \leq k$.

Proof. (1) Let $f$ be a $Q$-fuzzy set in $A$ defined by

$$f(x,q) = \begin{cases} \ 0 & \text{if } x \in I, \\ \ k & \text{if } x \notin I, \end{cases}$$

for all $q \in Q$.

Case 1. To show that $L(f;t) = I$ for all $t < k$, let $t \in [0,1]$ be such that $t < k$. Let $x \in L(f;t)$. Then $f(x,q) \leq t < k$ for all $q \in Q$. Thus $f(x,q) \neq k$ for all $q \in Q$, so $f(x,q) = 0$ for all $q \in Q$. Thus $x \in I$, so $L(f;t) \subseteq I$. Now, let $x \in I$. Then $f(x,q) = 0 \leq t$ for all $q \in Q$. Thus $x \in L(f;t)$, so $I \subseteq L(f;t)$. Hence, $L(f;t) = I$ for all $t < k$.

Case 2. To show that $L(f;t) = I$ for all $t \geq k$, let $t \in [0,1]$ be such that $t \geq k$. Clearly, $L(f;t) \subseteq A$. Let $x \in A$. Then

$$f(x,q) = \begin{cases} \ 0 < t & \text{if } x \in I, \\ \ k \leq t & \text{if } x \notin I, \end{cases}$$

for all $q \in Q$. Thus $x \in L(f;t)$, so $A \subseteq L(f;t)$. Hence, $L(f;t) = A$ for all $t \geq k$. We claim that $L(f;t,q) = L(f;t,q')$ for all $q,q' \in Q$. For $q,q' \in Q$, we obtain

$$x \in L(f;t,q) \iff f(x,q) \leq t \iff f(x,q') \leq t \iff x \in L(f;t,q').$$
Hence, $L(f; t; q) = L(f; t; q')$ for all $q, q' \in Q$. By Lemma 2.7 (1), we have $L(f; t) = \bigcap_{q \in Q} L(f; t; q)$. By the claim, we have $L(f; t) = L(f; t; q)$ for all $q \in Q$. Since $L(f; t; q) = I$ for all $t \leq k$ and $L(f; t; q) = A$ for all $t \geq k$, it follows from Theorem 2.9 (1) that $\mathcal{T}$ is a $Q$-fuzzy UP-ideal of $A$. By Remark 1.1, we have $L(\mathcal{T}; t) = L(f; t) = I$ for all $t < k$ and $L(\mathcal{T}; t) = L(f; t) = A$ for all $t \geq k$.

Let $\mathcal{T} = \mathcal{G}$. Then $\mathcal{G}$ is a $Q$-fuzzy UP-ideal of $A$ such that $L(\mathcal{G}; t) = I$ for all $t < k$ and $L(\mathcal{G}; t) = A$ for all $t \geq k$.

(2) Let $f$ be a $Q$-fuzzy set in $A$ defined by

\[
 f(x, q) = \begin{cases} 
 1 & \text{if } x \in I, \\
 k & \text{if } x \notin I,
\end{cases}
\]

for all $q \in Q$.

Case 1: To show that $U(f; t) = I$ for all $t < k$, let $t \in [0, 1]$ be such that $t < k$. Let $x \in U(f; t)$. Then $f(x, q) \geq t$ for all $q \in Q$. Thus $f(x, q) = k$ for all $q \in Q$, so $f(x, q) = 1$ for all $q \in Q$. Thus $x \notin I$, so $U(f; t) \subseteq I$. Now, let $x \in I$. Then $f(x, q) = 1 \geq t$ for all $q \in Q$. Thus $x \in U(f; t)$, so $I \subseteq U(f; t)$. Hence, $U(f; t) = I$ for all $t > k$.

Case 2: To show that $U(f; t) = A$ for all $t \leq k$, let $t \in [0, 1]$ be such that $t \leq k$. Clearly, $U(f; t) \subseteq A$. Let $x \in A$. Then

\[
 f(x, q) = \begin{cases} 
 k & \text{if } x \notin I, \\
 1 & \text{if } x \in I,
\end{cases}
\]

for all $q \in Q$. Thus $x \in U(f; t)$, so $A \subseteq U(f; t)$. Hence, $U(f; t) = A$ for all $t \leq k$.

We claim that $U(f; t; q) = U(f; t; q')$ for all $q, q' \in Q$. For $q, q' \in Q$, we obtain

\[
 x \in U(f; t; q) \iff f(x, q) \geq t \iff f(x, q') \geq t \iff x \in U(f; t; q').
\]

Hence, $U(f; t; q) = U(f; t; q')$ for all $q, q' \in Q$. By Lemma 2.7 (3), we have $U(f; t) = \bigcap_{q \in Q} U(f; t; q)$. By the claim, we have $U(f; t) = U(f; t; q)$ for all $q \in Q$. Since $U(f; t; q) = U(f; t) = I$ for all $t > k$ and $U(f; t; q) = U(f; t) = A$ for all $t \leq k$, it follows from Theorem 2.9 (3) that $f$ is a $Q$-fuzzy UP-ideal of $A$.

Corollary 2.14. Let $S$ be a UP-subalgebra of $A$. Then the following statements hold:

(1) for any $k \in (0, 1]$, there exists a $Q$-fuzzy UP-subalgebra $g$ of $A$ such that $L(\mathcal{G}; t) = S$ for all $t < k$ and $L(\mathcal{G}; t) = A$ for all $t \geq k$.

(2) for any $k \in [0, 1]$, there exists a $Q$-fuzzy UP-subalgebra $f$ of $A$ such that $U(f; t) = S$ for all $t > k$ and $U(f; t) = A$ for all $t \leq k$.

Proof. (1) Let $f$ be a $Q$-fuzzy set in $A$ defined by

\[
 f(x, q) = \begin{cases} 
 0 & \text{if } x \in S, \\
 k & \text{if } x \notin S,
\end{cases}
\]

for all $q \in Q$.

In the proof of Corollary 2.13 (1), we have $L(f; t) = S$ for all $t < k$ and $L(f; t) = A$ for all $t \geq k$, and $L(f; t; q) = L(f; t; q')$ for all $q, q' \in Q$. By Lemma 2.7 (1), we have $L(f; t) = \bigcap_{q \in Q} L(f; t; q)$.
Remark 1.17. We have $L(f; t) = L(f; t) = S$ for all $t < k$ and $L(f; t) = L(f; t) = A$ for all $t \geq k$. Let $\mathcal{F} = \{\mathcal{F} \}$, then $\mathcal{F}$ is a $Q$-fuzzy UP-subalgebra of $A$ such that $L(\mathcal{F}; t) = S$ for all $t < k$ and $L(\mathcal{F}; t) = A$ for all $t \geq k$.

(2) Let $f$ be a $Q$-fuzzy set in $A$ defined by

$$f(x, q) = \begin{cases} 1 & \text{if } x \in S, \\ k & \text{if } x \notin S, \end{cases}$$

for all $q \in Q$.

In the proof of Corollary 2.13 (2), we have $L(f; t) = S$ for all $t > k$ and $U(f; t) = A$ for all $t \leq k$, and $U(f; t, q) = U(f; t, q)$ for all $q \in \mathcal{Q}$. By Lemma 2.1 (3), we have $U(f; t) = \bigcap_{q \in \mathcal{Q}} U(f; t, q)$. By the claim, we have $U(f; t) = U(f; t, q)$ for all $q \in Q$. Since $U(f; t, q) = U(f; t) = S$ for all $t > k$ and $U(f; t, q) = U(f; t) = A$ for all $t \leq k$, it follows from Theorem 2.11 (3) that $f$ is a $Q$-fuzzy UP-subalgebra of $A$.

Theorem 2.15. Let $f$ be a $Q$-fuzzy set in $A$ and $s < t$, $s, t \in [0, 1]$. Then the following statements hold:

(1) $L(f; s, q) = L(f; t, q)$ if and only if there is no $x \in A$ such that $s < f(x, q) \leq t$,

(2) $L^-(f; s, q) = L^-(f; t, q)$ if and only if there is no $x \in A$ such that $s \leq f(x, q) < t$,

(3) $U(f; s, q) = U(f; t, q)$ if and only if there is no $x \in A$ such that $s < f(x, q) < t$,

(4) $U^+(f; s, q) = U^+(f; t, q)$ if and only if there is no $x \in A$ such that $s \leq f(x, q) \leq t$.

Proof. (1) Assume that $L(f; s, q) \neq L(f; t, q)$. Suppose that there is $x \in A$ such that $s < f(x, q) \leq t$. Then $x \in L(f; t, q)$ but $x \notin L(f; s, q)$, so $L(f; t, q) \neq L(f; s, q)$ which is a contradiction. Hence, there is no $x \in A$ such that $s < f(x, q) \leq t$.

Conversely, assume that there is no $x \in A$ such that $s < f(x, q) \leq t$. Let $x \notin L(f; s, q)$. Then $f(x, q) \leq s$. Thus $x \in L(f; t, q)$, so that $L(f; s, q) \subseteq L(f; t, q)$. Suppose that $L(f; t, q) \notin L(f; s, q)$. Then there exists $x \notin L(f; s, q)$ but $x \in L(f; t, q)$. Thus $f(x, q) \leq t$ and $f(x, q) > s$, so $s < f(x, q) \leq t$ which is a contradiction. Thus $L(f; t, q) \subseteq L(f; s, q)$. Hence, $L(f; s, q) = L(f; t, q)$.

(2) Similarly to as in the proof of (1).

(3) Assume that $U(f; s, q) = U(f; t, q)$. Suppose that there is $x \in A$ such that $s \leq f(x, q) < t$. Then $x \in U(f; s, q)$ but $x \notin U(f; t, q)$, so $U(f; s, q) \neq U(f; t, q)$ which is a contradiction. Hence, there is no $x \in A$ such that $s \leq f(x, q) < t$.

Conversely, assume that there is no $x \in A$ such that $s \leq f(x, q) < t$. Let $x \notin U(f; s, q)$. Then $f(x, q) \geq s$. Thus $x \in U(f; t; q)$, so $U(f; s, q) \subseteq U(f; t; q)$. Suppose that $U(f; s, q) \notin U(f; t; q)$. Then there exists $x \in U(f; s, q)$ but $x \notin U(f; t; q)$. Thus $f(x, q) \leq t$ and $s < f(x, q) < t$ which is a contradiction. Thus $U(f; s, q) \subseteq U(f; t; q)$. Hence, $U(f; s, q) = U(f; t; q)$.

(4) Similarly to as in the proof of (3).

Corollary 2.16. Let $f$ be a $Q$-fuzzy set in $A$ and $s < t$ for $s, t \in [0, 1]$. Then the following statements hold:

(1) $L(f; s, q) = L(f; t, q)$ if and only if $U^+(f; s, q) = U^+(f; t, q)$ and

(2) $L(f; s, q) = L(f; t, q)$ if and only if $L^-(f; s, q) = L^-(f; t, q)$.

Proof. (1) It follows from Theorem 2.15 (1) and Theorem 2.15 (4).

(2) It follows from Theorem 2.15 (2) and Theorem 2.15 (3).
Theorem 2.17. Let $(A; 0, 1)$ and $(B; s, 0, 1)$ be UP-algebras and let $f: A \rightarrow B$ be a UP-homomorphism. Then the following statements hold:

1. If $\mu$ is a $q$-fuzzy UP-ideal of $B$, then $\mu_f$ is also a $q$-fuzzy UP-ideal of $A$, and
2. If $\mu$ is a $q$-fuzzy UP-subalgebra of $B$, then $\mu_f$ is also a $q$-fuzzy UP-subalgebra of $A$.

Proof. (1) Assume that $\mu$ is a $q$-fuzzy UP-ideal of $B$. Let $x \in A$. Then

$$
\mu_f(0_A, q) = \mu_f(0_B, q) = \mu_f(0_B, q) > \mu_f(x, q) = \mu_f(x, q).$

Let $x, y, z \in A$. Then

$$
\mu_f(x \cdot y, q) = \mu(f(x \cdot y), q) = \mu(f(x) \cdot f(y), q) \\
\geq \min\{\mu(f(x) \cdot f(y), q), \mu(f(y), q)\} = \min\{\mu(f(x) \cdot f(y), q), \mu(f(y), q)\} \\
= \min\{\mu(f(x) \cdot f(y), q), \mu(f(y), q)\}.
$$

Hence, $\mu_f$ is a $q$-fuzzy UP-ideal of $A$.

(2) Assume that $\mu$ is a $q$-fuzzy UP-subalgebra of $B$. Let $x, y \in A$. Then

$$
\mu_f(x \cdot y, q) = \mu(f(x \cdot y), q) \\
\geq \min\{\mu(f(x), q), \mu(f(y), q)\}.
$$

Hence, $\mu_f$ is a $q$-fuzzy UP-subalgebra of $A$.

With Definition 1.10 and 1.13 and Theorem 2.17, we obtain the corollary.

Corollary 2.18. Let $f: A \rightarrow B$ be a UP-homomorphism. Then the following statements hold:

1. If $\mu$ is a $Q$-fuzzy UP-ideal of $B$, then $\mu_f$ is also a $Q$-fuzzy UP-ideal of $A$, and
2. If $\mu$ is a $Q$-fuzzy UP-subalgebra of $B$, then $\mu_f$ is also a $Q$-fuzzy UP-subalgebra of $A$.

Theorem 2.19. Let $(A; 0, 1)$ and $(B; s, 0, 1)$ be UP-algebras and let $f: A \rightarrow B$ be a UP-isomorphism. Then the following statements hold:

1. If $\mu_f$ is a $q$-fuzzy UP-ideal of $A$, then $\mu$ is also a $q$-fuzzy UP-ideal of $B$, and
2. If $\mu_f$ is a $q$-fuzzy UP-subalgebra of $A$, then $\mu$ is also a $q$-fuzzy UP-subalgebra of $B$.

Proof. (1) Assume that $\mu_f$ is a $q$-fuzzy UP-ideal of $A$. Let $y \in B$. Then there exists $x \in A$ such that $f(x) = y$, we have
\[
\begin{align*}
\mu([0, a], q) - \mu([y, 0], q) \\
= \mu(f(x) * f([0, a], q)) \\
= \mu(f(x) * f([0], q)) \\
= \mu(f(x) * [0, q]) \\
= \mu(f(x), q) \\
\geq \mu(f(x), q) \\
= \mu(f(x), q) \\
= \mu(y, q).
\end{align*}
\] (UP-3) (Proposition 1.22) (Definition 1.10 (1))

Let \(a, b, c \in B\). Then there exist \(x, y, z \in A\) such that \(f(x) = a\), \(f(y) = b\) and \(f(z) = c\), we have

\[
\begin{align*}
\mu([a, b, c], q) &= \mu([f(x), f(y), f(z)], q) \\
&= \mu([f(x), f(y), f(z)], q) \\
&= \mu([f(x), f(y), f(z)], q) \\
&= \min\{\mu([f(x), f(y), f(z)], q)\} \\
&= \min\{\mu([f(x), f(y), f(z)], q)\} \\
&= \min\{\mu([f(x), f(y), f(z)], q)\}.
\end{align*}
\] (Definition 1.10 (2)) (Definition 1.13)

Hence, \(\mu\) is a \(q\)-fuzzy UP-ideal of \(B\).

(2) Assume that \(\mu_f\) is a \(q\)-fuzzy UP-subalgebra of \(A\). Let \(a, b \in B\). Then there exist \(x, y \in A\) such that \(f(x) = a\) and \(f(y) = b\), we have

\[
\begin{align*}
\mu([a, b, c], q) &= \mu([f(x), f(y), f(z)], q) \\
&= \mu([f(x), f(y), f(z)], q) \\
&= \mu([f(x), f(y), f(z)], q) \\
&= \min\{\mu([f(x), f(y), f(z)], q)\} \\
&= \min\{\mu([f(x), f(y), f(z)], q)\} \\
&= \min\{\mu([f(x), f(y), f(z)], q)\}.
\end{align*}
\] (Definition 1.10 (2)) (Definition 1.13)

Hence, \(\mu\) is a \(q\)-fuzzy UP-subalgebra of \(B\).

With Definition 1.10 and 1.13 and Theorem 2.19, we obtain the corollary.

Corollary 2.20. Let \(f: A \rightarrow B\) be a UP-isomorphism. Then the following statements hold:

1. if \(\mu_f\) is a \(Q\)-fuzzy UP-ideal of \(A\), then \(\mu\) is also a \(Q\)-fuzzy UP-ideal of \(B\), and

2. if \(\mu_f\) is a \(Q\)-fuzzy UP-subalgebra of \(A\), then \(\mu\) is also a \(Q\)-fuzzy UP-subalgebra of \(B\).

Lemma 2.21. (Bali, 2005) For any \(a, b, c, d \in B\), the following properties hold:

1. \(\max\{\max\{a, b\}, \max\{c, d\}\} = \max\{\max\{a, c\}, \max\{b, d\}\}\), and

2. \(\min\{\min\{a, b\}, \min\{c, d\}\} = \min\{\min\{a, c\}, \min\{b, d\}\}\).

Let \((A, \cdot, 0_A)\) and \((B, \ast, 0_B)\) be UP-algebras. We can easily prove that \(A \times B\) is a UP-algebra defined by

\[
(x_1, x_2) \circ (y_1, y_2) = (x_1 \cdot y_1, x_2 \ast y_2)
\]

for all \(x_1, y_1 \in A\) and \(x_2, y_2 \in B\).
Theorem 2.22. Let \((A; \cdot, 0_A)\) and \((B; *, 0_B)\) be UP-algebras. Then the following statements hold:

1. If \(\mu\) is a \(q\)-fuzzy UP-ideal of \(A\) and \(\delta\) is a \(q\)-fuzzy UP-ideal of \(B\), then \(\mu \cdot \delta\) is a \(q\)-fuzzy UP-ideal of \(A \times B\), and

2. If \(\mu\) is a \(q\)-fuzzy UP-subalgebra of \(A\) and \(\delta\) is a \(q\)-fuzzy UP-subalgebra of \(B\), then \(\mu \cdot \delta\) is a \(q\)-fuzzy UP-subalgebra of \(A \times B\).

Proof. (1) Assume that \(\mu\) is a \(q\)-fuzzy UP-ideal of \(A\) and \(\delta\) is a \(q\)-fuzzy UP-ideal of \(B\). Let \((x_1, x_2) \in A \times B\). Then

\[
(\mu \cdot \delta)((0_A, 0_B), q) = \min\{\mu(0_A, q), \delta(0_B, q)\} \geq \min\{\mu(x_1, q), \delta(x_2, q)\} \quad \text{(Definition 1.10 (1))}
\]

Let \((x_1, x_2), (y_1, y_2), (z_1, z_2) \in A \times B\). Then

\[
(\mu \cdot \delta)((x_1, x_2) \circ (z_1, z_2), q) = (\mu \cdot \delta)((x_1 \cdot z_1, x_2 \ast z_2), q) \\
= \min\{\mu(x_1 \cdot z_1, q), \delta(x_2 \ast z_2, q)\} \geq \min\{\min\{\mu(x_1, q) \cdot \mu(y_1, q), \delta(y_2, q)\},\min\{\delta(x_2 \ast z_2, q), \delta(y_2, q)\}\} \quad \text{(Definition 1.10 (2))}
\]

\[
= \min\{\min\{\mu(x_1, q) \cdot \mu(y_1, q), \delta(y_2, q)\},\min\{\delta(x_2 \ast z_2, q), \delta(y_2, q)\}\} \quad \text{(Lemma 2.21 (2))}
\]

Hence, \(\mu \cdot \delta\) is a \(q\)-fuzzy UP-ideal of \(A \times B\).

(2) Assume that \(\mu\) is a \(q\)-fuzzy UP-subalgebra of \(A\) and \(\delta\) is a \(q\)-fuzzy UP-subalgebra of \(B\). Let \((x_1, x_2), (y_1, y_2) \in A \times B\). Then

\[
(\mu \cdot \delta)((x_1, x_2) \circ (y_1, y_2), q) = (\mu \cdot \delta)((x_1 \cdot y_1, x_2 \ast y_2), q) \\
= \min\{\mu(x_1 \cdot y_1, q), \delta(x_2 \ast y_2, q)\} \geq \min\{\min\{\mu(x_1, q) \cdot \mu(y_1, q), \delta(y_2, q)\},\min\{\delta(x_2 \ast y_2, q), \delta(y_2, q)\}\} \quad \text{(Definition 1.13)}
\]

\[
= \min\{\min\{\mu(x_1, q) \cdot \mu(y_1, q), \delta(y_2, q)\},\min\{\delta(x_2 \ast y_2, q), \delta(y_2, q)\}\} \quad \text{(Lemma 2.21 (2))}
\]

Hence, \(\mu \cdot \delta\) is a \(q\)-fuzzy UP-subalgebra of \(A \times B\).

Give examples of conflict that \(\mu\) and \(\delta\) are \(q\)-fuzzy UP-ideals (resp. \(q\)-fuzzy UP-subalgebras) of \(A\) but \(\mu \times \delta\) is not a \(q\)-fuzzy UP-ideal (resp. \(q\)-fuzzy UP-subalgebra) of \(A \times A\).

Example 2.23. Let \(A = \{0, 1\}\) be a set with a binary operation defined by the following Cayley table:

\[
\begin{array}{cc}
 0 & 1 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{array}
\]

Then \((A; \cdot, 0)\) is a UP-algebra. Let \(Q = \{q\}\). We define \(Q\)-fuzzy sets \(\mu\) and \(\delta\) in \(A\) as follows: \(\mu(0, q) = 0.2, \delta(0, q) = 0.3, \mu(1, q) = 0.1\) and \(\delta(1, q) = 0.1\). Using this data, we can show that \(\mu\) and \(\delta\) are \(q\)-fuzzy UP-ideals of \(A\). Let \((x_1, x_2) = (0, 0), (y_1, y_2) = (1, 0), (z_1, z_2) = (1, 1) \in A \times A\). Then
\[(\mu \times \delta)((x_1, x_2) \ominus (y_1, y_2), q) = 0.1\]

and

\[\min\{\mu \times \delta)((x_1, x_2) \ominus (y_1, y_2), (x_1, x_2) \ominus (y_1, y_2), q) = 0.2.\]

Hence, \((\mu \times \delta)((x_1, x_2) \ominus (y_1, y_2), q) \neq \min\{\mu \times \delta)((x_1, x_2) \ominus (y_1, y_2), q)\}

\((\mu \times \delta)((y_1, y_2), q)\). Therefore, \(\mu \times \delta\) is not a q-fuzzy UP-ideal of \(A \times A\).

**Example 2.24.** Let \(A = \{0, 1, 2\}\) be a set with a binary operation \(\cdot\) defined by the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Then \((A, \cdot, 0)\) is a UP-algebra. Let \(\mathcal{Q} = \{q\}\). We defined a q-fuzzy set \(\mu\) and \(\delta\) in \(A\) as follows: \(\mu(0, q) = 0.4, \delta(0, q) = 0.7, \mu(1, q) = 0.1, \delta(1, q) = 0.1, \mu(2, q) = 0.3\) and \(\delta(2, q) = 0.3\). Using this data, we can show that \(\mu\) and \(\delta\) are q fuzzy UP-subalgebras of \(A\). Let \((x_1, x_2) = (0, 1), (y_1, y_2) = (1, 2) \in A \times A\). Then

\[(\mu \times \delta)((x_1, x_2) \ominus (y_1, y_2), q) = 0.1\]

and

\[\min\{\mu \times \delta)((x_1, x_2), q), (\mu \times \delta)((y_1, y_2), q) = 0.3.\]

Hence, \((\mu \times \delta)((x_1, x_2) \ominus (y_1, y_2), q) \neq \min\{\mu \times \delta)((x_1, x_2), q), (\mu \times \delta)((y_1, y_2), q)\}.

Therefore, \(\mu \times \delta\) is not a q-fuzzy UP-subalgebra of \(A \times A\).

With Definition 1.10 and 1.13 and Theorem 2.22, we obtain the corollary.

**Corollary 2.25.** The following statements hold:

1. If \(\mu\) is a q-fuzzy UP-ideal of \(A\) and \(\delta\) is a q-fuzzy UP-ideal of \(B\), then \(\mu \cdot \delta\) is a q-fuzzy UP-ideal of \(A \times B\), and
2. If \(\mu\) is a q-fuzzy UP-subalgebra of \(A\) and \(\delta\) is a q-fuzzy UP-subalgebra of \(B\), then \(\mu \cdot \delta\) is a q-fuzzy UP-subalgebra of \(A \times B\).

**Theorem 2.26.** If \(\mu\) is a q-fuzzy set on \(A\) and \(\delta\) is a q-fuzzy set on \(B\) such that \(\mu \cdot \delta\) is a q-fuzzy UP-ideal of \(A \times B\), then the following statements hold:

1. either \(\mu(0_A, q) \geq \mu(x, q)\) for all \(x \in A\) or \(\delta(0_B, q) \geq \delta(x, q)\) for all \(x \in B\),
2. if \(\mu(0_A, q) \geq \mu(x, q)\) for all \(x \in A\), then either \(\delta(0_B, q) \geq \delta(x, q)\) for all \(x \in B\), and
3. if \(\delta(0_B, q) \geq \delta(x, q)\) for all \(x \in B\), then either \(\mu(0_A, q) \geq \mu(x, q)\) for all \(x \in A\) or \(\mu(0_A, q) \geq \delta(x, q)\) for all \(x \in A\).

**Proof.** (1) Suppose that there exist \(x \in A\) and \(y \in B\) such that \(\mu(0_A, q) \geq \mu(x, q)\) and \(\delta(0_B, q) \geq \delta(y, q)\). Then

\[(\mu \cdot \delta)((x, y), q) = \min\{\mu(x, q), \delta(y, q)\}
\]

which is a contradiction. Hence, \(\mu(0_A, q) \geq \mu(x, q)\) for all \(x \in A\) or \(\delta(0_B, q) \geq \delta(x, q)\) for all \(x \in B\).
(2) Assume that $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$. Suppose that there exist $x \in A$ and $y \in B$ such that $\delta(0_B, q) < \mu(x, q)$ and $\delta(x, q) < \delta(y, q)$. Then $\mu(0_A, q) \geq \mu(x, q) > \delta(0_B, q)$. Thus

$$
(\mu \cdot \delta)((x, y), q) = \min\{\mu(x, q), \delta(y, q)\}
> \min\{\delta(0_B, q), \delta(0_B, q)\}
= \delta(0_B, q)
= \min\{\mu(0_A, q), \delta(0_B, q)\}
= (\mu \cdot \delta)((0_A, 0_B), q)
$$

which is a contradiction. Hence, $\delta(0_B, q) \geq \mu(x, q)$ for all $x \in A$ or $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$.

(3) Assume that $\delta(0_B, q) > \delta(x, q)$ for all $x \in B$. Suppose that there exist $x \in A$ and $y \in B$ such that $\mu(0_A, q) < \mu(x, q)$ and $\mu(0_A, q) < \delta(y, q)$. Then $\delta(0_B, q) \geq \delta(x, q) > \mu(0_A, q)$. Thus,

$$
(\mu \cdot \delta)((x, y), q) = \min\{\mu(x, q), \delta(y, q)\}
> \min\{\mu(0_A, q), \mu(0_A, q)\}
= \mu(0_A, q)
= \min\{\mu(0_A, q), \delta(0_B, q)\}
= (\mu \cdot \delta)((0_A, 0_B), q)
$$

which is a contradiction. Hence, $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$ or $\mu(0_A, q) \geq \delta(x, q)$ for all $x \in B$.

With Definition 1.10 and 1.13 and Theorem 2.26, we obtain the corollary.

Corollary 2.27. If $\mu$ is a $Q$-fuzzy set in $A$ and $\delta$ is a $Q$-fuzzy set in $B$ such that $\mu \cdot \delta$ is a $Q$-fuzzy UP-ideal of $A \times B$, then the following statements hold:

(1) for all $q \in Q$, either $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$ or $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$,

(2) for all $q \in Q$, if $\mu(0_A, q) < \mu(x, q)$ for all $x \in A$, then either $\delta(0_B, q) < \mu(x, q)$ for all $x \in A$ or $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$; and

(3) for all $q \in Q$, if $\delta(0_B, q) < \delta(x, q)$ for all $x \in B$, then either $\mu(0_A, q) < \mu(x, q)$ for all $x \in A$ or $\mu(0_A, q) \geq \delta(x, q)$ for all $x \in B$.

Theorem 2.28. Let $(A; \ast, 0_A)$ and $(B; \ast, 0_B)$ be UP-algebras and let $\mu$ be a $Q$-fuzzy set in $A$ and $\delta$ be a $Q$-fuzzy set in $B$. Then the following statements hold:

(1) if $\mu \cdot \delta$ is a $Q$-fuzzy UP-ideal of $A \times B$, then either $\mu$ is a $Q$-fuzzy UP-ideal of $A$ or $\delta$ is a $Q$-fuzzy UP-ideal of $B$; and

(2) if $\mu \cdot \delta$ is a $Q$-fuzzy UP-subalgebra of $A \times B$, then either $\mu$ is a $Q$-fuzzy UP-subalgebra of $A$ or $\delta$ is a $Q$-fuzzy UP-subalgebra of $B$.
Proof. (1) Assume that $\mu \cdot \delta$ is a q-fuzzy UP ideal of $A \times B$. Suppose that $\mu$ is not a q-fuzzy UP ideal of $A$ and $\delta$ is not a q-fuzzy UP ideal of $B$. By Theorem 2.26 (1), we have $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$ or $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$. Suppose that $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$. By Theorem 2.26 (2), either $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in A$ or $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$. If $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in A$, then $(\mu \cdot \delta)(x, 0_B, q) = \min\{\mu(x, q), \delta(0_B, q)\} = \mu(x, q)$. We consider, for all $x, y, z \in A$,

$$\mu(x \cdot z, q) = \min\{\mu(x \cdot z, q), \delta(0_B, q)\} = (\mu \cdot \delta)((x \cdot z, 0_B), q) = (\mu \cdot \delta)((x \cdot z, 0_B \cdot 0_B), q) = \min\{\mu(x, q) \cdot z, 0_B, q\} \geq \min\{\mu(x, q) \cdot (y, 0_B) \cdot (z, 0_B), q\},$$

$$= \min\{\mu(x, q) \cdot (y, 0_B), q\} \geq \min\{\mu(x, q) \cdot (y, 0_B), q\} = \mu(x, q).$$

Hence, $\mu$ is a q-fuzzy UP ideal of $A$ which is a contradiction. Suppose that $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$. By Theorem 2.26 (3), either $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$ or $\mu(0_A, q) \geq \delta(x, q)$ for all $x \in B$. If $\mu(0_A, q) \geq \delta(x, q)$ for all $x \in B$, then $(\mu \cdot \delta)((0_A, x), q) = \min\{\mu(0_A, q), \delta(x, q)\} = \delta(x, q)$. We consider, for all $x, y, z \in B$,

$$\delta(x \cdot z, q) = \min\{\mu(0_A, q), \delta(x \cdot z, q)\} = (\mu \cdot \delta)((0_A, x \cdot z), q) = (\mu \cdot \delta)((0_A, x \cdot (0_A \cdot z)), q) = \min\{\mu(0_A, q) \cdot (y \cdot z), q\} \geq \min\{\mu(0_A, q) \cdot (y \cdot z), q\} = \delta(x, q).$$

Hence, $\delta$ is a q-fuzzy UP ideal of $B$ which is a contradiction. Since $\mu$ is not a q-fuzzy UP ideal of $A$ and $\delta$ is not a q-fuzzy UP ideal of $B$, we have $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$ and $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$. Let $x_1, x_2, x_3 \in A$ and $y_1, y_2, y_3 \in B$ be such that $\mu(x_1, y_1, q) < \min\{\mu(x_1, x_2, q), \mu(x_2, q)\}$ and $\delta(y_1, y_2, q) < \min\{\delta(y_1, y_2, q), \delta(y_2, q)\}; \delta(y_1, y_2, q).$
\[
\min \{ \min \{ \mu(x_1 \cdot x_2, q), \mu(x_2, q) \} \}, \min \{ \delta(y_1 \ast (y_2 \ast y_3), q), \delta(y_2, q) \} \}.
\]
Thus
\[
\min \{ \mu(x_1 \cdot x_2, q), \delta(y_1 \ast y_2, q) \} = \min \{ \mu(x_1 \cdot x_2, q), \delta(y_1 \ast y_2, q) \} = \min \{ \mu(x_1 \cdot x_2, q), \delta(y_1 \ast y_2, q) \}.
\]
(Definition 1.25)
\[
\min \{ \mu(x_1 \cdot x_2, q), \delta(y_2, q) \} \leq \min \{ \min \{ \mu(x_1 \cdot x_2, q), \delta(y_1 \ast y_2, q) \}, \delta(y_2, q) \}.
\]
(Definiation 1.25)
\[
\min \{ \mu(x_1 \cdot x_2, q), \delta(y_2, q) \} \leq \min \{ \min \{ \mu(x_1 \cdot x_2, q), \delta(y_1 \ast y_2, q) \}, \delta(y_2, q) \}.
\]
(Definiation 1.25)

It follows that \( \min \{ \mu(x_1 \cdot x_2, q), \delta(y_1 \ast y_2, q) \} \leq \min \{ \mu(x_1 \cdot x_2, q), \delta(y_2, q) \} \) which is a contradiction. Hence, \( \mu \) is a \( q \)-fuzzy UP-ideal of \( A \) or \( \delta \) is a \( q \)-fuzzy UP-ideal of \( B \).

(2) Assume that \( \mu \cdot \delta \) is a \( q \)-fuzzy UP-subalgebra of \( A \times B \). Suppose that \( \mu \) is not a \( q \)-fuzzy UP-subalgebra of \( A \) and \( \delta \) is not a \( q \)-fuzzy UP-subalgebra of \( B \). Then there exist \( x, y \in A \) and \( a, b \in B \) such that
\[
\mu(x \cdot y, q) < \min \{ \mu(x, q), \mu(y, q) \} \text{ and } \delta(a \ast b, q) < \min \{ \delta(a, q), \delta(b, q) \}.
\]
Then \( \min \{ \mu(x \cdot y, q), \delta(a \ast b, q) \} < \min \{ \min \{ \mu(x, q), \mu(y, q) \}, \min \{ \delta(a, q), \delta(b, q) \} \}. \)
Consider,
\[
\min \{ \mu(x \cdot y, q), \delta(a \ast b, q) \} = \min \{ \mu(x \cdot y, q), \delta(a \ast b, q) \} = \min \{ \mu(x \cdot y, q), \delta(a \ast b, q) \} = \min \{ \mu(x \cdot y, q), \delta(a \ast b, q) \}.
\]
(Definition 1.25)
\[
\min \{ \mu(x \cdot y, q), \delta(a \ast b, q) \} \leq \min \{ \min \{ \mu(x, q), \mu(y, q) \}, \min \{ \delta(a, q), \delta(b, q) \} \} \text{ which is a contradiction. Hence, } \mu \text{ is a } q \text{-fuzzy UP-subalgebra of } A \text{ or } \delta \text{ is a } q \text{-fuzzy UP-subalgebra of } B.
\]

Give examples of conflict that \( \mu \) and \( \delta \) are not \( Q \) fuzzy UP ideals (resp. \( Q \) fuzzy UP-subalgebras) of \( A \) but \( \mu \cdot \delta \) is a \( Q \)-fuzzy UP-ideal (resp. \( Q \)-fuzzy UP-subalgebra) of \( A \times A \).

Example 2.29. Let \( A = \{0, 1\} \) be a set with a binary operation \( \cdot \) defined by the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \( (A; \cdot, 0) \) is a UP-algebra. Let \( Q = \{a, b\} \). We define two \( Q \)-fuzzy sets \( \mu \) and \( \delta \) in \( A \) as follows:

<table>
<thead>
<tr>
<th></th>
<th>\mu</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>1</td>
<td>b</td>
</tr>
</tbody>
</table>

and

\[
\delta(a, q) = \begin{cases} 
0.1 & \text{if } q = 0.1 \\
0.3 & \text{if } q = 0.3 
\end{cases}
\]

and


<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$0.1$</td>
</tr>
<tr>
<td>$1$</td>
<td>$0.3$</td>
<td>$0.3$</td>
</tr>
</tbody>
</table>

Since $\mu(0, a) = 0.1 < 0.3 = \mu(1, a)$, we have $\mu(0, a) \not\geq \mu(1, a)$. Thus $\mu$ is not an $a$-fuzzy UP-ideal of $A$. Since $\delta(0, b) = 0.1 < 0.3 = \delta(1, b)$, we have $\delta(0, b) \not\geq \delta(1, b)$. Thus $\delta$ is not a $b$-fuzzy UP-ideal of $A$. Therefore, $\mu$ and $\delta$ are not $Q$-fuzzy UP-ideals of $A$. Using the above data, we can show that $\mu \cdot \delta$ is a $Q$-fuzzy UP-ideal of $A \times A$.

**Example 2.30.** Let $A = \{0, 1\}$ be a set with a binary operation - defined by the following table:

\[
\begin{array}{c|cc}
\cdot & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{array}
\]

Then $(A, \cdot, 0)$ is a UP-algebra. Let $Q = \{a, b\}$. We defined two $Q$-fuzzy sets $\mu$ and $\delta$ in $A$ as follows:

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$0.1$</td>
<td>$0.3$</td>
</tr>
<tr>
<td>$1$</td>
<td>$0.3$</td>
<td>$0.3$</td>
</tr>
</tbody>
</table>

and

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$0.1$</td>
<td>$0.2$</td>
</tr>
<tr>
<td>$1$</td>
<td>$0.3$</td>
<td>$0.3$</td>
</tr>
</tbody>
</table>

Since $\mu(1, a) = \mu(0, a) = 0.1 < 0.3 = \min\{0.3, 0.3\} = \min\{\mu(1, a), \mu(1, a)\}$, we have $\mu(1, c) \not\geq \min\{\mu(1, a), \mu(1, a)\}$. Thus $\mu$ is not an $a$-fuzzy UP-subalgebra of $A$. Since $\delta(1, b) = \delta(0, b) = 0.1 < 0.3 = \min\{0.3, 0.3\} = \min\{\delta(1, b), \delta(1, b)\}$, we have $\delta(1, b) \not\geq \min\{\delta(1, b), \delta(1, b)\}$. Thus $\delta$ is not a $b$-fuzzy UP-subalgebra of $A$. Therefore, $\mu$ and $\delta$ are not $Q$-fuzzy UP-subalgebras of $A$. By Example 2.29, we have $\mu \cdot \delta$ is a $Q$-fuzzy UP-ideal of $A \times A$. By Corollary 2.2, we have $\mu \cdot \delta$ is a $Q$-fuzzy UP-subalgebra of $A \times A$.

**Acknowledgements**

The authors wish to express their sincere thanks to the referees for the valuable suggestions which lead to an improvement of this paper.

**References**


