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Bipolar soft connected, bipolar soft disconnected and bipolar soft compact spaces

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Abstract

Bipolar soft topological spaces are mathematical expressions to estimate interpretation of data frameworks. Bipolar soft theory considers the core features of data granules. Bipolarity is important to distinguish between positive information which is guaranteed to be possible and negative information which is forbidden or surely false. Connectedness and compactness are the most important fundamental topological properties. These properties highlight the main features of topological spaces and distinguish one topology from another. Taking this into account, we explore the bipolar soft connectedness, bipolar soft disconnectedness and bipolar soft compactness properties for bipolar soft topological spaces. Moreover, we introduce the notion of bipolar soft disjoint sets, bipolar soft separation, and bipolar soft hereditary property and study on bipolar soft connected and disconnected spaces. By giving the detailed picture of bipolar soft connected and disconnected spaces we investigate bipolar soft compact spaces and derive some results related to this concept.

Keywords: bipolar soft topology, bipolar soft disjoint set, bipolar soft open set, bipolar soft closed set, bipolar soft connected space, bipolar soft disconnected space, bipolar soft compact space

1. Introduction

We create models of reality that are improvements of aspects of the genuine world. Lamentably these scientific models are excessively convoluted and we cannot locate the exact solutions. The vulnerability or instability of information while modeling issues in social sciences, medical sciences, artificial intelligence, engineering, natural sciences, etc., makes the utilization of conventional classical method successful. Therefore, traditional set theories, which were based on the crisp and exact case may not be completely suitable for taking care of issues of ambiguity/vagueness. To surpass these instabilities, the sorts of theory have been proposed (Gau et al., 1993; Pawlak, 1982; Zadeh, 1965). Yet, all these speculations have their natural troubles. The reason behind these troubles is, potentially, the insufficiency of the parameterization instrument of the theory as stated by Molodtsov (1999). Molodtsov (1999) popularized the idea of the theory of soft sets as a new, effective and stronger mathematical tool for dealing with instabilities, which is free from the above challenges. In his paper, he exhibited the crucial after effects of the new theory and effectively connected it to a few headings; for example game theory, Riemann integration, probability, smoothness of function, and so forth. Soft frameworks give an exceptionally broad system with the contribution of parameters. A lot of work on soft set theory and its application in different fields have been carried out by a number of researchers (Aktas et al., 2007; Ali et al., 2009, 2010, 2011; Jun, 2008; Jun et al., 2008; Maji et al., 2003; Shabir et al., 2009). If we review the history of soft topological spaces, the foundation of which was laid by Shabir et al. (2011), we find many remarkable authors following them (Aygunoglu et al., 2012; Cagman et al., 2011; Hussain et al., 2011, 2014; Khalil et al., 2015; Lin, 2013; Min, *Corresponding author.
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The concept of bipolar soft sets (a hybridization of the structure of soft sets and bipolarity) with its application in decision making was introduced and discussed in detail by Shabir et al. (2013) and studied exhaustively by Karaaslan et al. (2014). A bipolar soft set is acquired by viewing not only a precisely chosen set of parameters, but also an associated set of oppositely meaning parameters called the “not set of parameters”. Due to the quality of providing positive and negative aspects of information at a time, the idea of the bipolar soft set is gaining momentum among researchers. Hayat et al. (2015) applied the concept of bipolar soft sets to hemirings. Recently, Shabir and Bakhtawar initiated the study of bipolar soft topological spaces. They defined bipolar soft topology as a collection τ of bipolar soft sets over the universe U. Consequently, they defined basic notions of bipolar soft topological spaces, bipolar soft open and bipolar soft closed sets, bipolar soft subspace, bipolar soft closure, bipolar soft interior, bipolar soft neighborhood of a point and investigated their several properties. Further, Shabir and Bakhtawar explored and studied in detail bipolar soft separation axioms.

In the present study we initiated some new ideas in bipolar soft topological spaces such as bipolar soft connected spaces, bipolar soft disconnected spaces, bipolar soft compact spaces. Section 2, presents Preliminaries on basic concepts related to soft sets, soft topological spaces, bipolar soft sets and bipolar soft topological spaces. Section 3, is devoted towards the idea of bipolar soft disjoint sets, bipolar soft separation of a set, bipolar soft connected spaces, bipolar soft disconnected spaces, and bipolar soft hereditary property, and some examples are given for the better understanding of these ideas. Section 3, studies the concept of bipolar soft compact spaces and some results related to these concepts are exhibited. These newly defined ideas in bipolar soft topological spaces will hopefully promote the future work and studies to be held in the bipolar soft topology and can be applied effectively to cope with uncertainties.

2. Preliminaries

In this section, we recall some basic definitions and results related to bipolar soft sets, soft topological spaces and bipolar soft topological spaces. Let U be an initial universe, E be the set of parameters, ¬E be the not set of parameters. Let P(U) denotes the power set of U and A, B, C be non-empty subsets of E.

Definition 1. (Maji et al., 2003) Let E = {e1, e2, ..., en} be the set of parameters. The not set of E denoted by ¬E is defined by ¬E = {¬e1, ¬e2, ..., ¬en} where for all i, ¬ei = not ei.

Definition 2. (Shabir et al., 2013) A triplet (F, G, A) is called a bipolar soft set over U, where F and G are mappings, F : A → P(U) and G : ¬A → P(U) such that F(e) ∪ G(¬e) = ∅ (Empty set) for all e ∈ A.

Definition 3. (Shabir et al., 2013) For two bipolar soft sets (F, G, A) and (F', G', B) over a universe U, we say that (F, G, A) is a bipolar soft subset of (F', G', B), if,

1) A ⊆ B and
2) F(e) ⊆ F'(e) and G(¬e) ⊆ G'(¬e) for all e ∈ A.

This relation is denoted by (F, G, A) ⊆ (F', G', B).

Definition 4. (Shabir et al., 2013). Two bipolar soft sets (F, G, A) and (F', G', B) over the universe U are said to be equal if (F, G, A) is a bipolar soft subset of (F', G', B) and (F', G', B) is a bipolar soft subset of (F, G, A).

Definition 5. (Shabir et al., 2013). The complement of a bipolar soft set (F, G, A) is denoted by (F, G, A)' and is defined by (F, G, A)' = (F', G', A) where F' : A → P(U) and G' : ¬A → P(U) are given by F'(e) = G(¬e) and G'(¬e) = F(e) for all e ∈ A.

Definition 6. (Shabir et al., 2013). Extended Union of two bipolar soft sets (F, G, A) and (F', G', B) over the common universe U is the bipolar soft set (H, I, C) over U where C = A ∪ B and for all e ∈ C,

\[
H(e) = \begin{cases} 
F(e) & \text{if } e \in A - B \\
F'(e) & \text{if } e \in B - A \\
F(e) \cup F'(e) & \text{if } e \in A \cap B 
\end{cases}
\]

\[
I(¬e) = \begin{cases} 
G(¬e) & \text{if } ¬e \in (¬A) - (¬B) \\
G'(¬e) & \text{if } ¬e \in (¬B) - (¬A) \\
G(¬e) \cap G'(¬e) & \text{if } ¬e \in (¬A) \cap (¬B) 
\end{cases}
\]

We denote it by (F, G, A) ∪ (F', G', B) = (H, I, C)
Definition 7. (Shabir et al., 2013). Extended Intersection of two bipolar soft sets \((F, G, A)\) and \((F, G, B)\) over the common universe \(U\) is the bipolar soft set \((H, I, C)\) over \(U\) where \(C = A \cup B\) and for all \(e \in C\),

\[
H(e) = \begin{cases} 
F(e) & \text{if } e \in A - B \\
F_i(e) & \text{if } e \in B - A \\
F(e) \cap F_i(e) & \text{if } e \in A \cap B \\
\end{cases}
\]

\[
I(-e) = \begin{cases} 
G(-e) & \text{if } -e \in (-A) - (-B) \\
G_i(-e) & \text{if } -e \in (-B) - (-A) \\
G(-e) \cup G_i(-e) & \text{if } -e \in (-A) \cap (-B) \\
\end{cases}
\]

We denote it by \(((F, G, A) \cap (F, G, B)) = (H, I, C)\).

Definition 8. (Shabir et al., 2013). Restricted Union of two bipolar soft sets \((F, G, A)\) and \((F, G, B)\) over the common universe \(U\) is the bipolar soft set \((H, I, C)\) over \(U\) where \(C = A \cap B\) is non-empty and for all \(e \in C\),

\[
H(e) = F(e) \cup F_i(e) \quad \text{and} \quad I(-e) = G(-e) \cap G_i(-e).
\]

We denote it by \(((F, G, A) \cup (F, G, B)) = (H, I, C)\).

Definition 9. (Shabir et al., 2013). Restricted Intersection of two bipolar soft sets \((F, G, A)\) and \((F, G, B)\) over the common universe \(U\) is the bipolar soft set \((H, I, C)\) over \(U\) where \(C = A \cap B\) is non-empty and for all \(e \in C\),

\[
H(e) = F(e) \cap F_i(e) \quad \text{and} \quad I(-e) = G(-e) \cup G_i(-e).
\]

We denote it by \(((F, G, A) \cap (F, G, B)) = (H, I, C)\).

Remark 1. If we take \(A = B = E\) in Definitions 6, 7, 8, 9 then extended union coincides with restricted union and extended intersection coincides with restricted intersection.

Proposition 1. (Shabir et al., 2013). Let \((F, G, A)\) and \((F, G, B)\) be two bipolar soft sets over the common universe \(U\) with the universe set of parameters \(E\). Then the following are true

1) \(((F, G, A) \cup (F, G, B)) = (F, G, A) \cap (F, G, B)\),
2) \(((F, G, A) \cap (F, G, B)) = (F, G, A) \cup (F, G, B)\),
3) \(((F, G, A) \cup_R (F, G, B)) = (F, G, A) \cap_R (F, G, B)\),
4) \(((F, G, A) \cap_R (F, G, B)) = (F, G, A) \cup_R (F, G, B)\).

Definition 10. (Shabir et al., 2013). The Union of two bipolar soft sets \((F, G, A)\) and \((F, G, A)\) over the common universe \(U\) is the bipolar soft set \((H, I, A)\) over \(U\) where for all \(e \in E\),

\[
H(e) = F(e) \cup F_i(e) \quad \text{and} \quad I(-e) = G(-e) \cap G_i(-e).
\]

We denote it by \((F, G, A) \cap (F, G, A) = (H, I, A)\).

Definition 11. (Shabir et al., 2013). The Intersection of two bipolar soft sets \((F, G, A)\) and \((F, G, A)\) over the common universe \(U\) is the bipolar soft set \((H, I, A)\) over \(U\) where for all \(e \in A\),

\[
H(e) = F(e) \cap F_i(e) \quad \text{and} \quad I(-e) = G(-e) \cup G_i(-e).
\]

We denote it by \((F, G, A) \cap (F, G, A) = (H, I, A)\).

Definition 12. (Shabir et al., 2013). A bipolar soft set over \(U\) is said to be relative null bipolar soft set (with respect to the parameter set \(A\), denoted by \((\Phi, \tilde{u}, A)\), if for all \(e \in A, \Phi(e) = \phi\) and \(\tilde{u}(-e) = U\), for all \(-e \in -A\).

The relative null bipolar soft set with respect to the universe set of parameters \(E\) is called the null bipolar soft set over \(U\) and is denoted by \((\Phi, \tilde{u}, U)\).

A bipolar soft set \((F, G, E)\) over \(U\) is said to be a non null bipolar soft set if \(F(e) \neq \phi\) for some \(e \in E\).

Definition 13. (Shabir et al., 2013). A bipolar soft set over \(U\) is said to be relative absolute bipolar soft set (with respect to the parameter set \(A\), denoted by \((\nu, \Theta, A)\), if for all \(e \in A, \nu(e) = U\) and \(\Theta(-e) = \phi\), for all \(-e \in -A\).

The relative absolute bipolar soft set with respect to the universe set of parameters \(E\) is called the absolute bipolar soft set over \(U\) and is denoted by \((\nu, \Theta, E)\).
Obviously, a bipolar soft set \((F, G, E)\) over \(U\) is said to be a non absolute bipolar soft set over \(U\), if \(F(e) \neq U\) for some \(e \in E\).

**Definition 14.** (Shabir and Bakhtawar). Let \((F, G, E)\) be a bipolar soft set over \(U\) and \(u \in U\). Then \(u \in (F, G, E)\) if \(u \in F(e)\) for all \(e \in E\).

If \(u \in (F, G, E)\) then automatically \(u \not\in G(\neg e)\) for all \(\neg e \in \neg E\).

Note that for any \(u \in U\), \(u \not\in (F, G, E)\), if \(u \not\in F(e)\) for some \(e \in E\).

**Definition 15.** (Shabir and Bakhtawar). Let \(Y\) be a non-empty subset of \(U\). Then \((\bar{Y}, \Theta, E)\) denotes the bipolar soft set over \(U\) defined by \(\bar{Y}(e) = Y\) for all \(e \in E\) and \(\Theta(\neg e) = \emptyset\) for all \(\neg e \in \neg E\).

**Definition 16.** (Shabir and Bakhtawar). Let \(u \in U\). Then \((F_u, G_u, E)\) denotes the bipolar soft set over \(U\), defined by \(F_u(e) = \{u\}\) and \(G_u(\neg e) = U \setminus \{u\}\), for each \(e \in E\).

**Definition 17.** (Shabir and Bakhtawar). Let \(\tau\) be the collection of bipolar soft sets over \(U\) with \(E\) as the set of parameters. Then \(\tau\) is said to be a bipolar soft topology on \(U\) if

1. \((\Phi, \bar{u}, E), (u, \Theta, E)\) belongs to \(\tau\)
2. the union of any number of bipolar soft sets in \(\tau\) belongs to \(\tau\)
3. the intersection of any two bipolar soft sets in \(\tau\) belongs to \(\tau\)

Then \((U, \tau, E, \neg E)\) is called a bipolar soft topological space over \(U\) and the members of \(\tau\) are said to be bipolar soft open sets in \(U\).

**Proposition 2.** (Shabir and Bakhtawar). Let \((U, \tau, E, \neg E)\) be a bipolar soft topological space over \(U\). Then the collection \(\tau = \{F(e) \mid (F, G, E) \in \tau\}\) for each \(e \in E\), defines a topology on \(U\).

**Definition 18.** (Shabir and Bakhtawar). Let \((U, \tau, E, \neg E)\) be a bipolar soft topological space over \(U\) and \((F, G, E)\) be a bipolar soft set over \(U\). Then \((F, G, E)\) is said to be bipolar soft closed if and only if \((F, G, E)\) belongs to \(\tau\).

A bipolar soft set \((F, G, E)\) over \(U\) is said to be bipolar soft clopen if it is both a bipolar soft closed and a bipolar soft open set over \(U\).

**Definition 19.** (Shabir and Bakhtawar). Let \((F, G, E)\) be a bipolar soft set over \(U\) and \(Y\) be a non-empty subset of \(U\). Then the bipolar sub soft set of \((F, G, E)\) over \(Y\) denoted by \(\left( F^* G, E \right) \), is defined as follows

\[
F^*(e) = Y \cap F(e) \quad \text{and} \quad G^*(\neg e) = Y \cap G(\neg e), \quad \text{for each} \quad e \in E.
\]

**Proposition 3.** (Shabir and Bakhtawar). Let \((U, \tau, E, \neg E)\) be a bipolar soft topological space over \(U\) and \(Y\) be a non-empty subset of \(X\). Then \(\tau_Y = \{F^*(G^*) \mid (F, G, E) \in \tau\}\) is a bipolar soft topology on \(Y\).

**Theorem 1.** (Shabir and Bakhtawar). Let \((U, \tau, E, \neg E)\) be a soft topological space over \(U\) (Shabir et al., 2011). Then the collection \(\tau\) consisting of bipolar soft sets \((F, G, E)\) such that \((F, G, E) \in \tau\) and \(G(\neg e) = F^*(e) = U \setminus F(e)\) for all \(\neg e \in \neg E\), defines a bipolar soft topology over \(U\).

### 3. Bipolar Soft Connected and Bipolar Soft Disconnected Spaces

In this section, we discussed and explored one of the most important property of bipolar soft topological spaces called the bipolar soft connectedness and bipolar soft disconnectedness.

**Definition 20.** Two bipolar soft sets \((F_1, G_1, E), (F_2, G_2, E)\) are said to be bipolar soft disjoint if \(F_1(e) \cap F_2(e) = \emptyset\) for all \(e \in E\).

**Definition 21.** Let \((U, \tau, E, \neg E)\) be a bipolar soft topological space over \(U\). A bipolar soft separation of \((\nu, \Theta, E)\) is a pair \((F, G, E), (F_1, G_1, E)\) of non-null disjoint bipolar soft open sets over \(U\) such that \(F_1(e) \cup F_2(e) = U\) for all \(e \in E\).

**Definition 22.** A bipolar soft topological space \((U, \tau, E, \neg E)\) is said to be a bipolar soft disconnected space if there exists a bipolar soft separation of \((\nu, \Theta, E)\).

Further, \((U, \tau, E, \neg E)\) is said to be a bipolar soft connected space if and only if it is not a bipolar soft disconnected space.

**Example 1.** Let \(U = \{m_1, m_2, m_3, m_4\}\) be the universe set representing “markets”. Let \(E = \{e_1, e_2\}\) = \{hand embroidery dresses, formal dresses\} and \(\neg E = \{-e_1, -e_2\}\) = \{machine embroidery, causal dresses\}. Let \((F, G, E), (F_1, G_1, E)\) represents the preferences of markets for selection of clothes by two women. Then the bipolar soft topology over \(U\) generated
by \((F, G, E), (F, G, E)\) and \((F, G, E)\) are given by \(\tau = \{\Phi, \underline{u}, (v, \Theta, E), (F, G, E), (F, G, E)\}\) where \((F, G, E), (F, G, E), (F, G, E)\) are bipolar soft sets over \(U\) defined as follows:

\[
\begin{align*}
F_1(e_i) &= \{m_i, 2\} \quad \Rightarrow \quad F_1(e_i) = \{m_i, 2\} \\
F_2(e_i) &= \{m_i, 2\} \quad \Rightarrow \quad F_2(e_i) = \{m_i, 2\} \\
F_1(e_i) &= \{m_i, 2\} \quad \Rightarrow \quad F_1(e_i) = \{m_i, 2\} \\
G_1(-e_i) &= \phi \quad \Rightarrow \quad G_1(-e_i) = \phi \\
F_1(e_i) &= \phi \quad \Rightarrow \quad F_1(e_i) = \phi \\
F_2(e_i) &= \phi \quad \Rightarrow \quad F_2(e_i) = \phi \\
G_2(-e_i) &= \phi \quad \Rightarrow \quad G_2(-e_i) = \phi \\
F_1(e_i) &= \phi \quad \Rightarrow \quad F_1(e_i) = \phi \\
F_2(e_i) &= \phi \quad \Rightarrow \quad F_2(e_i) = \phi \\
G_2(-e_i) &= \phi \quad \Rightarrow \quad G_2(-e_i) = \phi \\
\end{align*}
\]

Then \((U, \tau, E, -E)\) is a bipolar soft disconnected space because \((F, G, E)\) and \((F, G, E)\) form a bipolar soft separation of \((v, \Theta, E)\).

**Example 2.** Let \(U = \{w_1, w_2, w_3\}\) be the universe set representing “wedding marquees”. Let \(E = \{e_1, e_2, e_3\} = \{\text{Expensive}, \text{best food service, ideal decoration facility}\}\) be the set of parameters and \(-E = \{-e_1, -e_2, -e_3\} = \{\text{cheap, average food service, poor decoration facility}\}\) be the not set of parameters. Let \((F, G, E), (F, G, E)\) represents the choices made by two different families for the selection of wedding marquees. Then \(\tau = \{\Phi, \underline{u}, (v, \Theta, E), (F, G, E), (F, G, E)\}\) is the bipolar soft topology over \(U\) generated by \((F, G, E), (F, G, E)\) where

\[
\begin{align*}
F_1(e_i) &= \{w_1, w_2, w_3\} \\
F_2(e_i) &= \{w_1, w_2, w_3\} \\
F_3(e_i) &= \{w_1, w_2, w_3\} \\
F_4(e_i) &= \{w_1, w_2, w_3\} \\
F_5(e_i) &= \{w_1, w_2, w_3\} \\
G_1(-e_i) &= \phi \\
G_2(-e_i) &= \phi \\
G_3(-e_i) &= \phi \\
G_4(-e_i) &= \phi \\
G_5(-e_i) &= \phi \\
\end{align*}
\]

We note that the bipolar soft topological space generated by \((F, G, E), (F, G, E)\) is a bipolar soft connected space because there does not exist a bipolar soft separation of \((v, \Theta, E)\).

**Proposition 4.** Let \((F, G, E)\) be a bipolar soft set. Then

1) \((F, G, E) \cup (F', G', E) = (H, I, E)\), where \(H(e) = F(e) \cup F'(e) \subseteq U\) for each \(e \in E\) and

\[
I(-e) = G(-e) \cap G'(e) = \phi \quad \text{for each} \quad -e \in -E.
\]

2) \((F, G, E) \cap (F', G', E) = (H, I, E)\), where \(H(e) = F(e) \cap F'(e) = \phi \quad \text{for each} \quad e \in E\) and

\[
I(e) = G(e) \cap G'(e) = \phi \quad \text{for each} \quad -e \in -E.
\]

Further \((F, G, E), (F', G', E)\) will always satisfy \(F(e) \cup F'(e) = G(e) \cup G'(e)\) for all \(e \in E\).

**Proof.** Straightforward.

**Theorem 2.** A bipolar soft topological space \((U, \tau, E, -E)\) is bipolar soft disconnected space if and only if there exist two bipolar soft closed sets \((F, G, E), (F, G, E)\) with \(G(e) \neq \phi, G(e) \neq \phi\) for some \(e \in E\), such that \(G_1(-e) \cup G_2(-e) = U\) for all \(-e \in -E\) and \(G_1(-e) \cap G_2(-e) = \phi\) for all \(-e \in -E\).

**Proof.** First, suppose \((U, \tau, E, -E)\) is a bipolar soft disconnected space. Then there exist a bipolar soft separation of \((v, \Theta, E)\). Let \((F, G, E)\) and \((H, I, E)\) forms a bipolar soft separation of \((v, \Theta, E)\). Then

\[
\begin{align*}
F(e) \cup H(e) = U & \quad \text{for all} \quad e \in E \\
F(e) \neq \phi & \quad \text{for some} \quad e \in E \\
H(e) \neq \phi & \quad \text{for some} \quad e \in E \\
F(e) \cap H(e) = \phi & \quad \text{for all} \quad e \in E
\end{align*}
\]

As \(F(e) = G'(e) = G'(e) = G'(e)\), therefore from equation (1) we have \(G'(e) \cup G'(e) = U\). From equation (1), (2) and (3) we have \(G'(e) \cap G'(e) = \phi\) for all \(-e \in -E\), where \(G(e) \neq \phi\) and \(I(e) \neq \phi\) for some \(-e \in -E\).

Further \((F, G, E)\) and \((H, I, E)\) are bipolar soft closed sets, since \((F, G, E)\) and \((H, I, E)\) belong to \(\tau\).
Conversely, suppose that there exist two bipolar soft closed sets \( (F_{1}, G_{1}, E), (F_{2}, G_{2}, E) \) with \( G_{1}(\neg e) \neq \emptyset, G_{2}(\neg e) \neq \emptyset \) for some \( e \in E \), such that \( G_{1}(\neg e) \cup G_{2}(\neg e) = U \) for all \( e \in E \) and \( G_{1}(\neg e) \cap G_{2}(\neg e) = \emptyset \) for all \( e \in E \).

Then \( (F_{1}, G_{1}, E) \) and \( (F_{2}, G_{2}, E) \) are bipolar soft open sets with \( F_{1}^{c}(e) = G_{1}(\neg e) \neq \emptyset \) and \( F_{2}^{c}(e) = G_{2}(\neg e) \neq \emptyset \) for some \( e \in E \) such that \( F_{1}^{c}(e) \cup F_{2}^{c}(e) = G_{1}(\neg e) \cup G_{2}(\neg e) = U \) for all \( e \in E \) and \( F_{1}^{c}(e) \cap F_{2}^{c}(e) = \emptyset \) for all \( e \in E \).

Therefore \( (F_{1}, G_{1}, E) \) and \( (F_{2}, G_{2}, E) \) form a bipolar soft separation of \( (v, \Theta, E) \). Thus \( (U, \tau, E, \neg E) \) is a bipolar soft disconnected space.

Remark 2. The union of two bipolar soft connected spaces over the same universe need not to be a bipolar soft connected space.

Example 3. Let \( U = \{h_{1}, h_{2}\}, E = \{e_{1}, e_{2}\}, \neg E = \{\neg e_{1}, \neg e_{2}\}, \tau_{1} = \{(\Phi, \tilde{u}, E), (v, \Theta, E), (F_{1}, G_{1}, E)\} \) and \( \tau_{1} = \{(\Phi, \tilde{u}, E), (v, \Theta, E), (F_{2}, G_{2}, E)\} \) where \( F_{1}(e_{1}) = U, F_{1}(e_{2}) = \emptyset \) and \( G_{1}(\neg e_{1}) = \emptyset, G_{1}(\neg e_{2}) = U, F_{2}(e_{1}) = \emptyset, F_{2}(e_{2}) = U \) and \( G_{2}(\neg e_{1}) = U, G_{2}(\neg e_{2}) = \emptyset \).

Then \( (U, \tau_{1}, E, \neg E) \) and \( (U, \tau_{2}, E, \neg E) \) are bipolar soft connected spaces over \( U \). But we note that \( \tau_{1} \cup \tau_{2} \) is not a bipolar soft connected space because \( (F_{1}, G_{1}, E), (F_{2}, G_{2}, E) \) form a bipolar soft separation of \( (\Phi, \tilde{u}, E) \) in \( \tau_{1} \cup \tau_{2} \).

Proposition 5. The intersection of two bipolar soft connected spaces over the same universe is a bipolar soft connected space.

Proof. Let \( (U, \tau_{1}, E, \neg E) \) and \( (U, \tau_{2}, E, \neg E) \) be two bipolar soft connected spaces. Suppose to the contrary that \( (U, \tau_{1} \cap \tau_{2}, E, \neg E) \) is not a bipolar soft connected space. Then there exist bipolar soft sets \( (F_{1}, G_{1}, E), (F_{2}, G_{2}, E) \) belonging to \( \tau_{1} \cup \tau_{2} \) which forms a bipolar soft separation of \( (\Phi, \tilde{u}, E) \) in \( (U, \tau_{1} \cap \tau_{2}, E, \neg E) \). Since \( (F_{1}, G_{1}, E), (F_{2}, G_{2}, E) \in \tau_{1} \cap \tau_{2} \), then \( (F_{1}, G_{1}, E), (F_{2}, G_{2}, E) \in \tau_{1} \) and \( (F_{1}, G_{1}, E), (F_{2}, G_{2}, E) \in \tau_{2} \). This implies that \( (F_{1}, G_{1}, E), (F_{2}, G_{2}, E) \) form a bipolar soft separation of \( (\Phi, \tilde{u}, E) \) in \( (U, \tau_{1}, E, \neg E) \) and \( (F_{1}, G_{1}, E), (F_{2}, G_{2}, E) \) form a bipolar soft separation of \( (\Phi, \tilde{u}, E) \) in \( (U, \tau_{2}, E, \neg E) \) which is a contradiction to given hypothesis. Thus \( (U, \tau_{1} \cap \tau_{2}, E, \neg E) \) is a bipolar soft connected space.

Remark 3. The intersection of two bipolar soft disconnected spaces over the same universe need not to be a bipolar soft disconnected space.

Example 4. Let \( U = \{h_{1}, h_{2}, h_{3}\}, E = \{e_{1}, e_{2}\}, \neg E = \{\neg e_{1}, \neg e_{2}\}, \tau_{1} = \{(\Phi, \tilde{u}, E), (v, \Theta, E), (F_{1}, G_{1}, E), (F_{2}, G_{2}, E), (F_{3}, G_{3}, E)\} \) and \( \tau_{2} = \{(\Phi, \tilde{u}, E), (v, \Theta, E), (H_{1}, I_{1}, E), (H_{2}, I_{2}, E), (H_{3}, I_{3}, E)\} \) where

\[
\begin{align*}
F_{1}(e_{1}) &= \{h_{1}\}, F_{1}(e_{2}) = \{h_{2}\} \text{ and } G_{1}(\neg e_{1}) = \{h_{1}\}, G_{1}(\neg e_{2}) = \{h_{2}\}; \\
F_{2}(e_{1}) &= \{h_{2}, h_{3}\}, F_{2}(e_{2}) = \{h_{3}\} \text{ and } G_{2}(\neg e_{1}) = \emptyset, G_{2}(\neg e_{2}) = \{h_{3}\}; \\
H_{1}(e_{1}) &= \{h_{1}\}, H_{1}(e_{2}) = \{h_{2}\} \text{ and } I_{1}(\neg e_{1}) = \{h_{1}\}, I_{1}(\neg e_{2}) = \{h_{2}\}; \\
H_{2}(e_{1}) &= \{h_{2}\}, H_{2}(e_{2}) = \{h_{3}\} \text{ and } I_{2}(\neg e_{1}) = \{h_{2}\}, I_{2}(\neg e_{2}) = \{h_{3}\}; \\
H_{3}(e_{1}) &= \{h_{3}\}, H_{3}(e_{2}) = \emptyset \text{ and } I_{3}(\neg e_{1}) = \{h_{3}\}, I_{3}(\neg e_{2}) = \emptyset.
\end{align*}
\]

Then \( (U, \tau_{1}, E, \neg E) \) and \( (U, \tau_{2}, E, \neg E) \) are bipolar soft disconnected spaces over \( U \).

Now, \( \tau_{1} \cap \tau_{2} = \{(\Phi, \tilde{u}, E), (v, \Theta, E)\} \). We note that \( (U, \tau_{1} \cap \tau_{2}, E, \neg E) \) is not a bipolar soft disconnected space because there do not exist any two non null disjoint bipolar soft open sets \( (F, G, E) \) and \( (H, I, E) \) belonging to \( \tau_{1} \cap \tau_{2} \) such that \( F(e) \cup H(e) = U \) for all \( e \in E \).

Proposition 6. The union of two bipolar soft disconnected spaces over the same universe is a bipolar soft disconnected space.

Proof. Straightforward.

Theorem 3. Let \( (U, \tau, E, \neg E) \) be a bipolar soft topological space over \( U \) and let the bipolar soft sets \( (F_{1}, G_{1}, E) \) and \( (F_{2}, G_{2}, E) \) form a bipolar soft separation of \( (v, \Theta, E) \). If \( (Y, \tau, E, \neg E) \) is a bipolar soft connected subspace of \( (U, \tau, E, \neg E) \) then \( Y \subseteq F_{1}(e) \) for all \( e \in E \) or \( Y \subseteq F_{2}(e) \) for all \( e \in E \).

Proof. Since \( (F_{1}, G_{1}, E) \) and \( (F_{2}, G_{2}, E) \) form a bipolar soft separation of \( (v, \Theta, E) \), we have

\[ U \cap (F_{1}(e) \cup F_{2}(e)) = U \text{ for each } e \in E \] (1)
\[ F(e) \neq \varphi \text{ for some } e \in E \] (2)
\[ F(e) \neq \varphi \text{ for some } e \in E \] (3)
\[ F(e) \cap F(e) = \varnothing \text{ for all } e \in E \] (4)
As \( Y \subseteq U \), we have \( (F, F, G, E) \) and \( (F', F', G, E) \) are open in \( (Y, \tau, E, \neg E) \).

From equation (1)
\[ Y \cap (F(e) \cup F(e)) = Y \text{ for each } e \in E. \]
This implies
\[ (Y \cap F(e)) \cup (Y \cap F(e)) = Y \text{ for all } e \in E. \] (5)

Also from equation (4)
\[ (Y \cap F(e)) \cap (Y \cap F(e)) = \varnothing \text{ for all } e \in E. \]

As \( (Y, \tau, E, \neg E) \) is connected, so either \( Y \cap F(e) = \emptyset \) for all \( e \in E \) or \( Y \cap F(e) = \emptyset \) for all \( e \in E \).

If \( Y \cap F(e) = \emptyset \) for all \( e \in E \) then from equation (5) \( Y \cap F(e) = Y \) for all \( e \in E \) and this implies \( Y \subseteq F(e) \) for all \( e \in E \).

If \( Y \cap F(e) = \emptyset \) for all \( e \in E \) then from equation (5) \( Y \cap F(e) = Y \) for all \( e \in E \) and this implies \( Y \subseteq F(e) \) for all \( e \in E \).

**Remark 4.** The converse of Theorem 3 does not hold in general.

**Example 5.** Let \( U = \{h_1, h_2, h_3, h_4\} \), \( E = \{e_1, e_2\} \), \( -E = \{-e_1, -e_2\} \), and \( \tau = \{(\Phi, \tilde{\Theta}), (\tilde{\Theta}, \Theta), (F, G, E), (F', G, E), (F, G, E), (F, G, E)\} \) where
\[ F(e_1) = \{h_1\}, F(e_2) = \{h_1\} \text{ and } G_1(-e_1) = \{h_1, h_2, h_3\}, G_1(-e_2) = \{h_2, h_3, h_4\}, \]
\[ F(e_1) = \{h_1\}, F(e_2) = \{h_1\} \text{ and } G_1(-e_1) = \{h_1, h_2, h_3\}, G_1(-e_2) = \{h_2, h_3, h_4\}, \]
\[ F(e_1) = \{h_1, h_2\}, F(e_2) = \{h_1, h_2\} \text{ and } G_1(-e_1) = \{h_1, h_2, h_3\}, G_1(-e_2) = \{h_2, h_3, h_4\}, \]
\[ F(e_1) = \{h_1, h_2, h_3\}, F(e_2) = \{h_1, h_2, h_3\} \text{ and } G_1(-e_1) = \{h_1, h_2, h_3\}, G_1(-e_2) = \{h_2, h_3, h_4\}, \]
\[ F(e_1) = \{h_1, h_2, h_3, h_4\}, F(e_2) = \{h_1, h_2, h_3, h_4\} \text{ and } G_1(-e_1) = \{h_1, h_2, h_3, h_4\}, G_1(-e_2) = \{h_2, h_3, h_4\}. \]

Then \( (U, \tau, E, \neg E) \) is a bipolar soft topological space over \( U \). Also, note that \( (F, G, E), (F', G, E) \) form a bipolar soft separation of \( (\tilde{\Theta}, \Theta, E) \).

Now let \( Y = \{h_1, h_2\} \) then \( \{h_1, h_2\} \) then \( \tau_Y = \{(\Phi, \tilde{\Theta}), (\tilde{\Theta}, \Theta), (F, G, E), (F', G, E), (F, G, E), (F, G, E)\} \) is a bipolar soft topology over \( Y \), where
\[ F(e_1) = \{h_1\}, F(e_2) = \{h_1\} \text{ and } G_1(-e_1) = \{h_1\}, G_1(-e_2) = \{h_2\}, \]
\[ F(e_1) = \{h_1\}, F(e_2) = \{h_1\} \text{ and } G_1(-e_1) = \{h_1\}, G_1(-e_2) = \{h_2\}, \]
\[ F(e_1) = \{h_1, h_2\}, F(e_2) = \{h_1, h_2\} \text{ and } G_1(-e_1) = \{h_1, h_2\}, G_1(-e_2) = \{h_2, h_3\}, \]
\[ F(e_1) = \{h_1, h_2, h_3\}, F(e_2) = \{h_1, h_2, h_3\} \text{ and } G_1(-e_1) = \{h_1, h_2, h_3\}, G_1(-e_2) = \{h_2, h_3, h_4\}, \]
\[ F(e_1) = \{h_1, h_2, h_3, h_4\}, F(e_2) = \{h_1, h_2, h_3, h_4\} \text{ and } G_1(-e_1) = \{h_1, h_2, h_3, h_4\}, G_1(-e_2) = \{h_2, h_3, h_4\}. \]

One can easily note that \( Y \subseteq F(e) \) for all \( e \in E \). But \( (Y, \tau, E, \neg E) \) is not a bipolar soft connected space because \( (F, G, E) \) and \( (F', G, E) \) form a bipolar soft separation of \( (\tilde{\Theta}, \Theta, E) \).

**Remark 5.** If there exist a non null, non-absolute bipolar soft clopen set over \( U \), then \( (U, \tau, E, \neg E) \) need not be a bipolar soft disconnected space.

**Proposition 7.** Let \( (U, \tau, E, \neg E) \) be a bipolar soft topological space over \( U \). If there exist a non-null, non-absolute bipolar soft clopen set \( (F, G, E) \) over \( U \) such that \( F(e) \cup F'(e) = U \) for each \( e \in E \) then \( (U, \tau, E, \neg E) \) is a bipolar soft
disconnected space.

**Proof.** Since \((F, G, E)\) is a bipolar soft clopen set over \(U\) then \((F, G, E)\) is a bipolar soft clopen set over \(U\). Now, by given hypothesis and by proposition 4, we have \(F(e)\cup F'(e) = U\) for all \(e \in E\) and \(G(\neg e)\cap G'(\neg e) = \emptyset\) for each \(\neg e \in \neg E\) and \(F(e)\cap F'(e) = e\) for each \(e \in E\) and \(G(e)\cup G'(e) = U\) for each \(e \in E\).

Therefore, \((F, G, E)\) and \((F, G, E)\) form a bipolar soft separation of \((v, \Theta, E)\).

Thus \((U, \tau, E, \neg E)\) is a bipolar soft disconnected space.

**Example 6.** Let \(U = \{h_1, h_2, h_3\}, E = \{e_1, e_2\}, \neg E = \{\neg e_1, \neg e_2\}\) and \(\tau = \{\emptyset, \Phi, (v, \Theta, E), (F_1, G_1, E), (F_2, G_2, E), (F_3, G_3, E), (F_4, G_4, E)\}\) where \(F_1(e_1) = \{h_1\}, F_1(e_2) = \{h_2\}\) and \(G_1(\neg e_1) = \{h_2\}\), \(G_1(\neg e_2) = \{h_1\}\), \(F_2(e_1) = \{h_2\}, F_2(e_2) = \{h_1\}\) and \(G_2(\neg e_1) = \{h_1\}\), \(G_2(\neg e_2) = \{h_2\}\), \(F_3(e_1) = \{h_1, h_2\}, F_3(e_2) = \{h_3\}\) and \(G_3(\neg e_1) = \{h_3\}\), \(G_3(\neg e_2) = \emptyset\), \(F_4(e_1) = \emptyset, F_4(e_2) = \emptyset\) and \(G_4(\neg e_1) = \{h_1, h_2\}\), \(G_4(\neg e_2) = U\).

Then \((U, \tau, E, \neg E)\) is a bipolar soft topological space over \(U\). Note that \((F_1, G_1, E)\) is a non-null, non-absolute bipolar soft clopen set over \(U\) but \((U, \tau, E, \neg E)\) is not a bipolar soft disconnected space since there do not exist a bipolar soft separation of \((v, \Theta, E)\).

**Theorem 4.** Let \((U, \tau, E)\) be a soft topological space over \(U\) and \((U, \tau_s, E, \neg E)\) be a bipolar soft topological space over \(U\) constructed from \((U, \tau, E)\) as in Theorem db. If \((U, \tau, E)\) is a soft disconnected space (Hussain, 2014) over \(U\) then \((U, \tau_s, E, \neg E)\) is a bipolar soft disconnected space over \(U\).

**Proof.** Since \((U, \tau, E)\) is a soft disconnected space, therefore there exist non-null soft open sets (say) \((F, E)\), \((H, E)\) over \(U\) such that \(U = (F, E) \cup (H, E)\) and \((F, E) \cap (H, E) = \emptyset\).

Further, \((F, G, E)\), \((H, I, E)\) are non-null bipolar soft open sets (because \(F(e) \neq \emptyset\), \(H(e) \neq \emptyset\)) where for all \(\neg e \in \neg E\), \(G(\neg e) = U \setminus F(e)\) and \(I(\neg e) = U \setminus H(e)\) since \((F, E)\), \((H, E)\) belongs to \(\tau_s\).

Now, \(F(e) \cup H(e) = U\) for all \(e \in E\) and \(G(\neg e) \cap I(\neg e) = (U \setminus F(e)) \cap (U \setminus H(e)) = \emptyset\) for each \(e \in E\). Also, \(F(e) \cap H(e) = \emptyset\) for each \(e \in E\) and \(G(\neg e) \cup I(\neg e) = (U \setminus F(e)) \cup (U \setminus H(e)) = U\) for each \(e \in E\). This implies that \((F, G, E)\), \((H, I, E)\) belonging to \(\tau_s\), forms a bipolar soft separation of \((v, \Theta, E)\). Thus \((U, \tau_s, E, \neg E)\) is a bipolar soft disconnected space.

**Proposition 8.** Let \((Y, \tau', E, \neg E)\) and \((Z, \tau^*, E, \neg E)\) be two bipolar soft subspaces of \((U, \tau, E, \neg E)\) and let \(Y \subseteq Z\). Then \((Y, \tau', E, \neg E)\) is a bipolar soft subspace of \((Z, \tau^*, E, \neg E)\).

**Proof.** As \(Y \subseteq Z\) so \(Y = Y \cap Z\). Moreover each bipolar soft open set \((\tau', F, 2, G, E)\) of \((Y, \tau', E, \neg E)\) is of the form \(F(\tau') = Y \cap F(e)\) and \(G(\tau') = Y \cap G(e)\) for all \(e \in E\) where \((F, G, E)\) is a bipolar soft open set of \((U, \tau, E, \neg E)\).

Now for each \(e \in E\),
\[Y \cap F(e) = (Y \cap Z) \cap F(e)\] and \[Y \cap G(e) = (Y \cap Z) \cap G(e)\].
\[\Rightarrow Y \cap F(e) = Y \cap (Z \cap F(e))\] and \[Y \cap G(e) = Y \cap (Z \cap G(e))\].
\[\Rightarrow Y \cap F(e) = Y \cap \tau' F(e)\] and \[Y \cap G(e) = Y \cap \tau' G(e)\] where \((\tau', F, 2, G, E)\) is a bipolar soft open set in \((Z, \tau^*, E, \neg E)\).
**Theorem 5.** Let \( \{ (Y_a, \tau_a, E, \neg E) \}_{a \in J} \) be the collection of bipolar soft connected subspaces of a bipolar soft topological space \((U, \tau, E, \neg E)\). If \( \cap_{a \in J} Y_a \neq \emptyset \), then \( \left( \bigcup_{a \in J} Y_a, \tau_{\cup_{a \in J}} E, \neg E \right) \) is a bipolar soft connected subspace of \((U, \tau, E, \neg E)\).

**Proof.** Let \( \{ (Y_a, \tau_a, E, \neg E) \}_{a \in J} \) be a collection of bipolar soft connected subspaces of \((U, \tau, E, \neg E)\), such that \( \cap_{a \in J} Y_a \neq \emptyset \). Suppose that \( Y = \bigcup_{a \in J} Y_a \) and \( (Y, \tau, E, \neg E) \) be a disconnected subspace of \((U, \tau, E, \neg E)\). Let \( (F_i, G_i, E) \), \( (F_i, G_i, E) \) be a bipolar soft separation of \((Y, \Theta, E)\). Then

\[
Y \cap F_i(e) \neq \emptyset \quad \text{for some } e \in E. \tag{1}
\]

\[
Y \cap F_i(e) \neq \emptyset \quad \text{for some } e \in E. \tag{2}
\]

\[
(Y \cap F_i(e)) \cap (Y \cap F_i(e)) = Y \cap (F_i(e) \cap F_i(e)) = \emptyset \quad \text{for all } e \in E. \tag{3}
\]

Consider a fixed \( Y_a \). Then from equation (1)

\[
Y_a \cap (F_i(e) \cup F_i(e)) = Y_a \quad \text{for all } e \in E.
\]

From equation (4)

\[
Y_a \cap (F_i(e) \cap F_i(e)) = \emptyset \quad \text{for all } e \in E.
\]

Since \( (Y_a, \tau_{\cup_{a \in J}} E, \neg E) \) is a bipolar soft connected subspace of \((U, \tau, E, \neg E)\) so either \( Y_a \cap F_i(e) = \emptyset \) for all \( e \in E \) or \( Y_a \cap F_i(e) = \emptyset \) for all \( e \in E \).

Now there are three cases:

(i) \( Y_a \cap F_i(e) = \emptyset \) for all \( e \in E \) and for all \( \alpha \in J \)

(ii) \( Y_a \cap F_i(e) = \emptyset \) for all \( e \in E \) and for all \( \alpha \in J \)

(iii) for some \( \alpha \in J \), \( Y_a \cap F_i(e) = \emptyset \) and for other some \( \alpha \in J \), \( Y_a \cap F_i(e) = \emptyset \).

Case: (i)

If \( Y_a \cap F_i(e) = \emptyset \) for all \( e \in E \) and for all \( \alpha \in J \), then \( \left( \bigcup_{a \in J} Y_a \right) \cap F_i(e) = \emptyset \), that is \( Y \cap F_i(e) = \emptyset \) for all \( e \in E \).

This contradicts equation (2).

Case: (ii)

If \( Y_a \cap F_i(e) = \emptyset \) for all \( e \in E \) and for all \( \alpha \in J \), then \( \left( \bigcup_{a \in J} Y_a \right) \cap F_i(e) = \emptyset \), that is \( Y \cap F_i(e) = \emptyset \) for all \( e \in E \).

This contradicts equation (3).

Case: (iii)

As \( \cap_{a \in J} Y_a = \emptyset \), so there exist some \( x \in Y_a \) for all \( \alpha \in J \). Now by equation (1) \( x \in F_i(e) \cap F_i(e) \) for all \( e \in E \), this implies \( x \in F_i(e) \) or \( x \in F_i(e) \).

If \( x \in F_i(e) \) then \( Y_a \cap F_i(e) = \emptyset \) and if \( x \in F_i(e) \) then \( Y_a \cap F_i(e) = \emptyset \). So the case (iii) is not possible.

Hence our supposition is wrong and \((Y, \tau, E, \neg E)\) is a bipolar soft connected subspace of \((U, \tau, E, \neg E)\).

**Definition 23.** A property \( P \) of a bipolar soft topological space \((U, \tau, E, \neg E)\) is said to be a bipolar soft hereditary iff every bipolar soft subspace \((Y, \tau_Y, E, \neg E)\) of \((U, \tau, E, \neg E)\) also possesses the property \( P \).

**Remark 6.** The bipolar soft connectedness (respect. bipolar soft disconnected) is not a bipolar soft hereditary property.

**Example 7.** Let \( U = \{ h_1, h_2, h_3 \}, E = \{ e_1, e_2 \}, \neg E = \{ -e_1, -e_2 \} \) and \( \tau = \{ (\Phi, \tilde{u}, E), (\Theta, E), (F_i, G_i, E), (F_2, G_2, E), (F_3, G_3, E) \} \) where
Then \((U, \tau, E, -E)\) is a bipolar soft connected space.

Now take \(Y = \{h, h_1\}\). Then \(\tau_Y = \{(\Phi, \hat{Y}, E), (\hat{Y}, \Theta, E), (\tau, \hat{Y}, E), (\tau, \hat{Y}, G, E), (\tau, \hat{Y}, F, G, E)\}\), where

\[
'F_1(e_1) = \{h_1\}, 'F_1(e_2) = \{h\} \quad \text{and} \quad 'G_1(-e_1) = \{h_2, h_3\}, 'G_1(-e_2) = \{h_2, h_3\}.
\]

\[
'F_2(e_1) = \{h_1, h_2\}, 'F_2(e_2) = \{h_1, h_2\} \quad \text{and} \quad 'G_2(-e_1) = \{h_3, h_4\}, 'G_2(-e_2) = \{h_3, h_4\}.
\]

\[
'F_3(e_1) = \{h_1, h_2, h_3\}, 'F_3(e_2) = \{h_1, h_2, h_3\} \quad \text{and} \quad 'G_3(-e_1) = \varphi, 'G_3(-e_2) = \{h_3\}.
\]

\[
'F_1(e_1) = Y, 'F_1(e_2) = Y \quad \text{and} \quad 'G_1(-e_1) = \varphi, 'G_1(-e_2) = \varphi. \quad \text{Then} \quad (Y, \tau_Y, E, -E) \quad \text{is a bipolar soft disconnected subspace of bipolar soft connected space.}
\]

**Example 8.** Let \(U = \{h, h_1, h_3\}, E = \{e_1, e_2\}, -E = \{e_1, e_2\} \) and \(\tau = \{\Phi, \hat{U}, E), (\hat{U}, \Theta, E), (\tau, \hat{U}, E), (\tau, \hat{U}, G, E), (\tau, \hat{U}, F, G, E)\}\), \((F_1, G, E)\) where

\[
F_1(e_1) = \{h_1\}, F_1(e_2) = \{h\} \quad \text{and} \quad G_1(-e_1) = \{h_2\}, G_1(-e_2) = \{h_2, h_3\},
\]

\[
F_2(e_1) = \{h_1, h_2\}, F_2(e_2) = \{h_1, h_2\} \quad \text{and} \quad G_2(-e_1) = \{h_3\}, G_2(-e_2) = \{h_3\},
\]

\[
F_3(e_1) = \varphi, F_3(e_2) = \varphi \quad \text{and} \quad G_3(-e_1) = \{h_3\}, G_3(-e_2) = \{h_3\}.
\]

Then \((U, \tau, E, -E)\) is a bipolar soft connected space.

Now take \(Y = \{h_1\}\). Then \(\tau_Y = \{(\Phi, \hat{Y}, E), (\hat{Y}, \Theta, E), (\tau, \hat{Y}, E), (\tau, \hat{Y}, G, E), (\tau, \hat{Y}, F, G, E)\}\) where

\[
'F_1(e_1) = \varphi, 'F_1(e_2) = \varphi \quad \text{and} \quad 'G_1(-e_1) = \varphi, 'G_1(-e_2) = Y, \]

\[
'F_2(e_1) = Y, 'F_2(e_2) = Y \quad \text{and} \quad 'G_2(-e_1) = \varphi, 'G_2(-e_2) = \varphi, \]

\[
'F_3(e_1) = \varphi, 'F_3(e_2) = \varphi \quad \text{and} \quad 'G_3(-e_1) = \varphi, 'G_3(-e_2) = \varphi. \quad \text{Then} \quad (Y, \tau_Y, E, -E) \quad \text{is a bipolar soft connected subspace of bipolar soft disconnected space.}
\]

## 4. Bipolar Soft Compact Spaces

In this section, we study another important property of bipolar soft topological spaces called the bipolar soft compactness. Bipolar soft compact spaces are investigated and some results related to this concept are derived.

**Definition 24.** A family \(\Psi = \{(F_\alpha, G_\alpha, E)\}_{\alpha \in A}\) of bipolar soft sets is called the bipolar soft cover of a bipolar soft set \((F, G, E)\) if \((F, G, E) \subseteq \bigcup_{\alpha \in A}(F_\alpha, G_\alpha, E)\).

Further, it is called the bipolar soft open cover of a bipolar soft set \((F, G, E)\) if each member of \(\Psi\) is a bipolar soft open set. A bipolar soft subcover of \(\Psi\) is a subfamily of \(\Psi\) which is also a bipolar soft cover.

**Definition 25.** A bipolar soft topological space \((U, \tau, E, -E)\) is called a bipolar soft compact space, if each bipolar soft open cover of \((U, \Theta, E)\) has a finite bipolar soft subcover.

**Example 9.** Let \(U = \{h_1, h_2, h_3\}, E = \{e_1, e_2\}, -E = \{e_1, -e_2\}, \) and \(\tau = \{(\Phi, \hat{U}, E), (\hat{U}, \Theta, E), (F_1, G_1, E), (F_2, G_2, E), (F_3, G_3, E), (F_4, G_4, E), (F_5, G_5, E), (F_6, G_6, E)\}\) where \((F_1, G_1, E), (F_2, G_2, E), (F_3, G_3, E), (F_4, G_4, E), (F_5, G_5, E), (F_6, G_6, E)\) are bipolar soft sets over \(U\) defined as follows:

\[
F_1(e_1) = \{h_1\}, F_1(e_2) = \{h\} \quad \text{and} \quad G_1(-e_1) = \{h_2\}, G_1(-e_2) = \{h_2\},
\]

\[
F_2(e_1) = \{h_1, h_2\}, F_2(e_2) = \{h_1, h_2\} \quad \text{and} \quad G_2(-e_1) = \{h_3\}, G_2(-e_2) = \{h_3\},
\]

\[
F_3(e_1) = \{h_1, h_2, h_3\}, F_3(e_2) = \{h_1, h_2, h_3\} \quad \text{and} \quad G_3(-e_1) = \varphi, G_3(-e_2) = \varphi,
\]

\[
F_4(e_1) = \{h_1, h_2\}, F_4(e_2) = \{h_1, h_2\} \quad \text{and} \quad G_4(-e_1) = \varphi, G_4(-e_2) = \varphi.
\]
$F_i(e_i) = \varphi, F_i(e_i) = \varphi$ and $G_i(\neg e_i) = \{d_i\}, G_i(\neg e_i) = \{h_i\}$,
$F_i(e_i) = \{h_i\}, F_i(e_i) = \{h_i\}$ and $G_i(\neg e_i) = \varphi, G_i(\neg e_i) = \varphi$. Then $(U, \tau, E, E)$ is a bipolar soft topological space over $U$. Further, we can easily observe that $(U, \tau, E, E)$ is a bipolar soft compact space because every open cover of $(\nu, \Theta, E)$ has a finite subcover.

**Example 10.** Let $U = \mathbb{N}$ be the universe set of natural numbers, let $E = \{e_i, e_j\}$ and $E = \{-e_i, -e_j\}$ be the set of parameters and the not set of parameters, respectively. Let $\tau$ be the bipolar soft topology over $N$, consisting of all bipolar soft sets defined on the parameter set $E$, generated by the bipolar soft sets $(F_i, G_i, E), (F_e, G_e, E), (F_j, G_j, E), \ldots$, where
$F_i(e_i) = \{1, 2\}, F_i(e_i) = \{1, 2\}$ and $G_i(\neg e_i) = U \setminus \{1, 2\}, G_i(\neg e_i) = U \setminus \{1, 2\}$,
$F_i(e_i) = \{2, 3\}, F_i(e_i) = \{2, 3\}$ and $G_i(\neg e_i) = U \setminus \{2, 3\}, G_i(\neg e_i) = U \setminus \{2, 3\}$,
$F_i(e_i) = \{3, 4\}, F_i(e_i) = \{3, 4\}$ and $G_i(\neg e_i) = U \setminus \{3, 4\}, G_i(\neg e_i) = U \setminus \{3, 4\}$, \ldots,
$F_i(e_i) = \{n, n + 1\}, F_i(e_i) = \{n, n + 1\}$ and $G_i(\neg e_i) = U \setminus \{n, n + 1\}, G_i(\neg e_i) = U \setminus \{n, n + 1\}$

Then the bipolar soft topological space $(U, \tau, E, E)$ over $U$ generated by the bipolar soft sets $(F_i, G_i, E)\ x \in \mathbb{N}$ is not a bipolar soft compact space since $\Psi = \{(F_i, G_i, E)\ x \in \mathbb{N}\}$ is a bipolar soft open cover of $\mathbb{N}$ with no finite bipolar soft subcover.

**Definition 26.** Let $(U, \tau, E, E)$ and $(U, \tau, E, E)$ be two bipolar soft topological spaces over the universe $U$. If $\tau_i \subseteq \tau_j$, then $\tau_i$ is said to be finer than $\tau_j$. If $\tau_i \subseteq \tau_j$ or $\tau_j \subseteq \tau_i$, then $\tau_i$ is comparable with $\tau_j$.

**Proposition 9.** Let $(U, \tau, E, E)$ be a bipolar soft compact space and $\tau_i \subseteq \tau_j$. Then $(U, \tau, E, E)$ is a bipolar soft compact.

**Proof.** Let $\{(F_i, G_i, E)\}_{i \in J}$ be the bipolar soft open cover of $(\nu, \Theta, E)$ in $(U, \tau, E, E)$. Since $\tau_i \subseteq \tau_j$, then $\{(F_i, G_i, E)\}_{i \in J}$ is the bipolar soft open cover of $(\nu, \Theta, E)$ by bipolar soft open sets of $(U, \tau, E, E)$. But $(U, \tau, E, E)$ is a bipolar soft compact space.

Therefore $(U, \Theta, E) \subseteq (F_i, G_i, E) \cup (F_i, G_i, E) \ldots (F_i, G_i, E)$, for some $\alpha_i, \alpha_j, \ldots, \alpha_n \in J$. Hence $(U, \tau, E, E)$ is a bipolar soft compact space.

**Theorem 6.** Let $(Y, \tau, E, E)$ be a bipolar soft subspace of $(U, \tau, E, E)$. Then $(Y, \tau, E, E)$ is a bipolar soft compact space if and only if every cover of $(Y, \Theta, E)$ by bipolar soft open sets in $U$ contains a finite subcover.

**Proof.** Let $(Y, \tau, E, E)$ be a bipolar soft compact space and $\{(F_i, G_i, E)\}_{i \in J}$ be a cover of $(Y, \Theta, E)$ by bipolar soft open sets in $U$. Now, $Y \subseteq \cup_{i \in J} (Y \cap F_i(e))$ for each $e \in E$, and $Y \subseteq \cap_{i \in J} (Y \cap G_i(\neg e))$ for each $\neg e \in E$.

Therefore, $\{(Y \cap F_i, G_i, E)\}_{i \in J}$ is a bipolar soft open cover of $(Y, \Theta, E)$.

Since $(Y, \tau, E, E)$ is a bipolar soft compact space, therefore, we have $(Y, \Theta, E) \subseteq (Y \cap F_i, G_i, E) \cup \ldots \cup (Y \cap F_i, G_i, E)$ for some $\alpha_1, \alpha_2, \ldots, \alpha_n \in J$.

This implies that $\{(Y \cap F_i, G_i, E)\}_{i \in J}$ is a bipolar subcover of $(Y, \Theta, E)$ by bipolar soft open sets in $U$.

Conversely, suppose $\{(Y \cap F_i, G_i, E)\}_{i \in J}$ is a bipolar soft open cover of $(Y, \Theta, E)$. It is easy to see that $\{(F_i, G_i, E)\}_{i \in J}$ is a bipolar soft open cover of $(Y, \Theta, E)$ by bipolar soft open sets in $U$. Therefore, by given hypothesis we have $(Y, \Theta, E) \subseteq (F_i, G_i, E) \cup (F_i, G_i, E) \cup \ldots \cup (F_i, G_i, E)$ for some $\alpha_1, \alpha_2, \ldots, \alpha_n \in J$. Thus, $\{(Y \cap F_i, G_i, E)\}_{i \in J}$.
is a bipolar subcover of \( (Y, \Theta, E) \). Hence \( (Y, \tau_y, E, \neg E) \) is a bipolar soft compact space.

**Definition 27.** Let \( (U, \tau, E, \neg E) \) be a bipolar soft topological space over \( U \) and let \( \beta \subseteq \tau \). If every element of \( \tau \) can be written as the union of the elements of \( \beta \), then \( \beta \) is called a bipolar soft basis for bipolar soft topology \( \tau \). Each element of \( \beta \) is called a bipolar soft basis element.

**Example 11.** Let \( U = \{h_1, h_2, h_3, h_4\} \), \( E = \{e_1, e_2\} \), \( \neg E = \{-e_1, -e_2\} \), and \( \tau = \{(\Phi, \tilde{u}, E), (\nu, \Theta, E), (F_1, G_1, E), (F_2, G_2, E), (F_3, G_3, E), (F_4, G_4, E), (F_5, G_5, E), (F_6, G_6, E), (F_7, G_7, E), (F_8, G_8, E)\} \) where \( (F_1, G_1, E), (F_2, G_2, E), (F_3, G_3, E), (F_4, G_4, E), (F_5, G_5, E), (F_6, G_6, E), (F_7, G_7, E), (F_8, G_8, E) \) are bipolar soft sets over \( U \) defined as follows:

\[
F_1(e_1) = \{h_1, h_2, h_3\}, \quad F_1(e_2) = \{h_1, h_2\} \quad \text{and} \quad G_1(-e_1) = \{h_3\}, \quad G_1(-e_2) = \{h_3, h_4\}, \\
F_2(e_1) = \{h_1, h_2, h_4\}, \quad F_2(e_2) = \{h_1, h_2\} \quad \text{and} \quad G_2(-e_1) = \{h_3\}, \quad G_2(-e_2) = \{h_3, h_4\}, \\
F_3(e_1) = \{h_1, h_3, h_4\}, \quad F_3(e_2) = \emptyset \quad \text{and} \quad G_3(-e_1) = \emptyset, \quad G_3(-e_2) = U, \\
F_4(e_1) = \{h_1, h_2\}, \quad F_4(e_2) = \{h_1, h_2\} \quad \text{and} \quad G_4(-e_1) = \{h_3, h_4\}, \quad G_4(-e_2) = \{h_3, h_4\}, \\
F_5(e_1) = \{h_1, h_3, h_4\}, \quad F_5(e_2) = \emptyset \quad \text{and} \quad G_5(-e_1) = \emptyset, \quad G_5(-e_2) = U, \\
F_6(e_1) = \{h_1, h_2, h_3, h_4\}, \quad F_6(e_2) = \emptyset \quad \text{and} \quad G_6(-e_1) = \emptyset, \quad G_6(-e_2) = U, \\
F_7(e_1) = \{h_1, h_2\}, \quad F_7(e_2) = \{h_1, h_2\} \quad \text{and} \quad G_7(-e_1) = \emptyset, \quad G_7(-e_2) = \{h_5, h_4\}, \\
F_8(e_1) = \{h_1, h_3, h_4\}, \quad F_8(e_2) = \emptyset \quad \text{and} \quad G_8(-e_1) = \emptyset, \quad G_8(-e_2) = \{h_5, h_4\}.
\]

Then \( (U, \tau_y, E, \neg E) \) is a bipolar soft topological space over \( U \). Now if we take \( \beta = \{(\Phi, \tilde{u}, E), (F_1, G_1, E), (F_2, G_2, E), (F_3, G_3, E), (F_4, G_4, E), (F_5, G_5, E), (F_6, G_6, E), (F_7, G_7, E), (F_8, G_8, E)\} \), then \( \beta \) is a bipolar soft basis for \( \tau \).

Next, if we take \( \beta' \subseteq \tau \) where \( \beta' = \{(\Phi, \tilde{u}, E), (F_1, G_1, E), (F_2, G_2, E), (F_3, G_3, E), (F_4, G_4, E), (F_5, G_5, E), (F_6, G_6, E), (F_7, G_7, E), (F_8, G_8, E)\} \) then \( \beta' \) is not a bipolar soft basis because \( (\nu, \Theta, E) \) cannot be written as the union of the elements of \( \beta' \).

**Theorem 7.** A bipolar soft topological space \( (U, \tau, E, \neg E) \) is a bipolar soft compact space if and only if there is a bipolar soft basis \( \beta \) for \( \tau \) such that every bipolar soft cover of \( (\nu, \Theta, E) \) by the elements of \( \beta \) has a finite bipolar soft subcover.

**Proof.** Let \( (U, \tau, E, \neg E) \) be a bipolar soft compact space. Obviously \( \tau \) is a bipolar soft basis for \( \tau \). Therefore, every bipolar soft cover of \( (\nu, \Theta, E) \) by elements of \( \tau \) has a finite bipolar soft subcover.

Conversely, to show \( (U, \tau, E, \neg E) \) is a bipolar soft compact space, let \( \{L_{\alpha}, M_{\alpha}, E\}_{\alpha \in J} \) be a bipolar soft open cover of \( (\nu, \Theta, E) \). We can write \( L_{\alpha}, M_{\alpha}, E \) as a union of basis element for each \( \alpha \in J \). These elements form a bipolar soft open cover of \( (\nu, \Theta, E) \) such that \( \left\{ (F_{\beta}, G_{\beta}, E) \right\}_{\beta \in I} \). Now, by given hypothesis, for some \( \beta_1, \beta_2, \ldots, \beta_n \in I \), we have

\[
U = F_{\beta_1}(e) \cup F_{\beta_2}(e) \cup F_{\beta_3}(e) \cup \cdots \cup F_{\beta_n}(e), \quad \text{for each } e \in E \\
\varphi = G_{\beta_1}(-e) \cap G_{\beta_2}(-e) \cap \cdots \cap G_{\beta_n}(-e) \quad \text{for each } -e \in \neg E.
\]

That is \( (\nu, \Theta, E) = (F_{\beta_1}, G_{\beta_1}, E) \cup (F_{\beta_2}, G_{\beta_2}, E) \cup \cdots \cup (F_{\beta_n}, G_{\beta_n}, E) \) for some \( \beta_1, \beta_2, \ldots, \beta_n \in I \). Now, \( F_{\beta_i}, G_{\beta_i}, E \subseteq L_{\alpha_i}, M_{\alpha_i}, E \), for each \( 1 \leq i \leq n \). This implies that \( \left\{ L_{\alpha_i}, M_{\alpha_i}, E \right\}_{i=1}^n \) is a finite bipolar subcover of \( (\nu, \Theta, E) \). Hence \( (U, \tau, E, \neg E) \) is a bipolar soft compact space.
5. Conclusions

During the study we have gone into detail about defining and finding out the properties of bipolar soft connected spaces, bipolar soft disconnected spaces and bipolar soft compact spaces. The credit of strengthening the foundations in the tool box of bipolar soft topology will be given to these newly defined concepts. The findings and results which we have drawn can be applied to solve existing problems in various fields which contain uncertainty.

References
