Original Article

Spectrum and fine spectrum of the lower triangular matrix $B(r,s,t)$ over the sequence space $cs$

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Abstract

Fine spectra of various matrix operators on different sequence spaces have been examined by several authors. Recently, some authors have determined the approximate point spectrum, the defect spectrum and the compression spectrum of various matrix operators on different sequence spaces. Here in this article we have determined the spectrum and fine spectrum of the lower triangular matrix $B(r,s,t)$ on the sequence space $cs$. In a further development, we have also determined the approximate point spectrum, the defect spectrum and the compression spectrum of the operator $B(r,s,t)$ on the sequence space $cs$.

Keywords: spectrum of an operator, matrix mapping, sequence space

1. Introduction

By $w$, we denote the space of all real or complex valued sequences. Throughout the article $c$, $c_0$, $bv$, $bs$, $\ell_1$, $\ell_\infty$ represent the spaces of all convergent, null, bounded variation, bounded series, absolutely summable and bounded sequences respectively. Also $bv_0$ denotes the sequence space $bv \cap c_0$.

Fine spectra of various matrix operators on different sequence spaces have been examined by several authors. The spectrum and fine spectrum of the Zweier Matrix on the sequence space $\ell_1$ and $bv$ was studied by Altay and Karakuş (2005). Altay and Başar (2004, 2005) determined the fine spectrum of the difference operator $\Delta$ and the generalized difference operator $B(r,s,t)$ on the sequence spaces $c$ and $c$. Furkan et al. (2006) have determined the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces $\ell_p$ and $bv$. Altun (2011a, 2011b) determined the fine spectrum of triangular Toeplitz operators and tridiagonal symmetric matrices over some sequence spaces. The fine spectra of the Cesàro operator $C$ over the sequence space $bv_0, (1 \leq p < \infty)$ was determined by Akhmedov and Başar (2008). Okutoyi (1990) determined the spectrum of the Cesàro operator $C_1$ on the sequence space $bv_0$. Fine spectra of operator $B(r,s,t)$ over the sequence spaces $\ell_1$ and $bv$ and generalized difference operator $B(r,s)$ over the sequence spaces $\ell_p$ and $bv_0, (1 \leq p < \infty)$, were studied by Bilgiç and Furkan (2007, 2008). Fine spectrum of the generalized difference operator $\Delta_1$ on the sequence space $\ell_1$ was investigated by Srivastava and Kumar (2010). Panigrahi and Srivastava (2011, 2012) studied the spectrum and fine spectrum of the second order difference operator $\Delta_2$ on the sequence space $c_0$ and $c$ was studied by Karakaya and Altun (2010). Karaisa and Başar (2013) have determined the spectrum and fine spectrum of the upper triangular matrix $A(r,s,t)$ over the sequence space $\ell_p, (0 < p < \infty)$. In a further development, they have also...
determined the approximate point spectrum, defect spectrum and compression spectrum of the operator \( A(r,s,t) \) on the sequence space \( \ell_p \) \((0 < p < \infty)\). The approximate point spectrum, defect spectrum and compression spectrum of the operator \( B(r,s) \) on the sequence spaces \( e_r, c_r, \ell_r \) and \( b_r \) \((1 < p < \infty)\) were studied by Başar, Durna and Yıldırım (2011).

The notion of matrix transformations over sequence space has been studied from various aspects. Besides the above listed workers, the spectrum and fine spectrum for various matrix operators has been investigated by Tripathy and Das (2014, 2015), Tripathy and Pal (2013a, 2013b, 2014), Tripathy and Saikia (2013) and many others in recent years.

In this paper, we will determine the spectrum and fine spectrum of the lower triangular matrix \( B(r,s,t) \) on the sequence space \( cs \). Also, we will determine the approximate point spectrum, the defect spectrum and the compression spectrum of the operator \( B(r,s,t) \) on the sequence space \( cs \).

Clearly, \( cs = \{x = (x_r) \in w : \lim_{n \to \infty} \sum_{i=0}^{n} x_r \text{ exists} \} \) is a Banach space with respect to the norm \( \|x\|_r = \sup \left\{ \sum_{i=0}^{n} |x_r| \right\} \).

2. Preliminaries and Background

Let \( X \) and \( Y \) be Banach spaces and \( T : X \to Y \) be a bounded linear operator. By \( R(T) \), we denote the range of \( T \), i.e.

\[
R(T) = \{ y \in Y : \exists x \in X \text{ such that } T(x) = y \}.
\]

By \( B(X) \), we denote the set of all bounded linear operators on \( X \) itself. If \( T \in B(X) \), then the adjoint \( T' \) of \( T \) is a bounded linear operator on the dual \( X' \) of \( X \) defined by \( (T'f)(x) = f(Tx) \), for all \( f \in X' \) and \( x \in X \).

Let \( X \neq \{0\} \) be a complex normed linear space and \( T : D(T) \to X \) be a linear operator with domain \( D(T) \subseteq X \). With \( T \) we associate the operator

\[
T_{\lambda} = T - \lambda I,
\]

where \( \lambda \) is a complex number and \( I \) is the identity operator on \( D(T) \). If \( T_{\lambda} \) has an inverse which is linear, we denote it by \( T_{\lambda}^{-1} \), that is

\[
T_{\lambda}^{-1} = (T - \lambda I)^{-1},
\]

call it the resolvent operator of \( T \).

Let \( X \neq \{0\} \) be a complex normed linear space and \( T : D(T) \to X \) be a linear operator with domain \( D(T) \subseteq X \). A regular value \( \lambda \) of \( T \) is a complex number such that

(1) \( T_{\lambda}^{-1} \) exists,

(2) \( T_{\lambda}^{-1} \) is bounded,

(3) \( T_{\lambda}^{-1} \) is defined on a set which is dense in \( X \) i.e.

\[
R(T_{\lambda}) = X.
\]

The resolvent set of \( T \), denoted by \( \rho(T,X) \), is the set of all regular values \( \lambda \) of \( T \). Its complement \( \sigma(T,X) = \mathbb{C} \setminus \rho(T,X) \) in the complex plane \( \mathbb{C} \) is called the spectrum of \( T \). Furthermore, the spectrum \( \sigma(T,X) \) is partitioned into three disjoint sets as follows:

The point (discrete) spectrum \( \sigma_p(T,X) \) is the set such that \( T_{\lambda}^{-1} \) does not exist. Any such \( \lambda \in \sigma_p(T,X) \) is called an eigenvalue of \( T \).

The continuous spectrum \( \sigma_c(T,X) \) is the set such that \( T_{\lambda}^{-1} \) exists and satisfies (R3), but not (R2), that is, \( T_{\lambda}^{-1} \) is unbounded.

The residual spectrum \( \sigma_r(T,X) \) is the set such that \( T_{\lambda}^{-1} \) exists (and may be bounded or not), but does not satisfy (R3), that is, the domain of \( T_{\lambda}^{-1} \) is not dense in \( X \).

If \( X \) is a Banach space and \( T \in B(X) \), then there are three possibilities for \( R(T) \) and \( T^{-1} \):

(1) \( R(T) = X \),

(2) \( R(T) \neq R(T) = X \),

(3) \( R(T) \neq X \),

and

(1) \( T^{-1} \) exists and is continuous,

(2) \( T^{-1} \) exists but is discontinuous,

(3) \( T^{-1} \) does not exist.

(One may refer to Goldberg (1985))

Applying Goldberg’s classification to \( T_{\lambda} \), we have three possibilities for \( T_{\lambda} \) and \( T_{\lambda}^{-1} \):

(1) \( T_{\lambda} \) is injective and \( T_{\lambda}^{-1} \) is continuous,

(2) \( T_{\lambda} \) is injective but \( T_{\lambda}^{-1} \) is discontinuous,

(3) \( T_{\lambda} \) is not injective.

If these possibilities are combined in all possible ways, nine different states are created which may be shown as in Table 1.

These are labeled by: \( I_1, I_2, I_3, II_1, II_2, III_1, III_2 \) and \( \lambda \). If \( \lambda \) is a complex number such that \( T_{\lambda} \in I_1 \) or \( T_{\lambda} \in II_2 \), then \( \lambda \) is in the resolvent set \( \rho(X) \) of \( T \). The further classification gives rise to the fine spectrum of \( T \). If an operator is in state \( II_1 \) for example, then \( R(T) \neq R(T) = X \) and \( T^{-1} \) exists but is discontinuous and we write \( \lambda \in II_1 \sigma(T,X) \).

Again, following Appell et al. (2004), we define the three more subdivisions of the spectrum called as the approximate point spectrum, defect spectrum and compression spectrum.

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
<th>III</th>
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<tbody>
<tr>
<td>1</td>
<td>( \rho(T,X) )</td>
<td>( \sigma_p(T,X) )</td>
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<tr>
<td>2</td>
<td>( \sigma_c(T,X) )</td>
<td>( \sigma_p(T,X) )</td>
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<td>3</td>
<td>( \sigma_r(T,X) )</td>
<td>( \sigma_p(T,X) )</td>
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Given a bounded linear operator \( T \) in a Banach space \( X \), we call a sequence \( (x_k) \) in \( X \) as a Weyl sequence for \( T \) if 
\[
\|x_k\| = 1 \quad \text{and} \quad \|Tx_k\| \to 0 \quad \text{as} \quad k \to \infty.
\]

The approximate point spectrum of \( T \), denoted by \( \sigma_{ap}(T, X) \), is defined as the set
\[
\sigma_{ap}(T, X) = \{ \lambda \in \mathbb{C} : \exists \text{a Weyl sequence for } T - \lambda I \}.
\] (2.1)

The defect spectrum of \( T \), denoted by \( \sigma_s(T, X) \), is defined as the set
\[
\sigma_s(T, X) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not surjective} \}.
\] (2.2)

The two subspectra given by (2.1) and (2.2) form a (not necessarily disjoint) decomposition
\[
\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_s(T, X)
\] (2.3)

of the spectrum. There is another subspectrum, \( \sigma_{co}(T, X) = \{ \lambda \in \mathbb{C} : R(T - \lambda I) \neq X \} \) which is often called the compression spectrum of \( T \). The compression spectrum gives rise to another (not necessarily disjoint) decomposition
\[
\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_{co}(T, X)
\] (2.4)

Clearly, \( \sigma_s(T, X) \subseteq \sigma_{ap}(T, X) \) and \( \sigma_{co}(T, X) \subseteq \sigma_s(T, X) \).

Moreover, it is easy to verify that
\[
\sigma_s(T, X) = \sigma_{ap}(T, X) \setminus \sigma_s(T, X) \quad \text{and}
\]
\[
\sigma_{co}(T, X) = \sigma(T, X) \setminus \left[ \sigma_s(T, X) \cup \sigma_{co}(T, X) \right]
\]

By the definitions given above, we can illustrate the subdivisions spectrum in Table 2.

**Proposition 2.1.** [Appell et al. (2004), Proposition 1.3, p.28]: Spectra and subspectra of an operator \( T \in B(X) \) and its adjoint \( T^* \in B(X^*) \) are related by the following relations:

(a) \( \sigma(T^*, X^*) = \sigma(T, X) \).

(b) \( \sigma_c(T^*, X^*) \subseteq \sigma_{ap}(T, X) \).

(c) \( \sigma_{ap}(T^*, X^*) = \sigma_s(T, X) \).

(d) \( \sigma_{co}(T^*, X^*) = \sigma_{co}(T, X) \).

(e) \( \sigma_s(T^*, X^*) = \sigma_s(T, X) \).

(f) \( \sigma_{co}(T^*, X^*) = \sigma_{co}(T, X) \).

(g) \( \sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_{co}(T^*, X^*) \)

\[
= \sigma_s(T, X) \cup \sigma_{co}(T^*, X^*)
\]

The relations (c)-(f) show that the approximate point spectrum is in a certain sense dual to defect spectrum, and the point spectrum dual to the compression spectrum.

The equality (g) implies, in particular, that \( \sigma(T, X) = \sigma_{ap}(T, X) \) if \( X \) is a Hilbert space and \( T \) is normal.

Roughly speaking, this shows that normal (in particular, self-adjoint) operators on Hilbert spaces are most similar to matrices in finite dimensional spaces (Appell et al., 2004).

Let \( \lambda \) and \( \mu \) be two sequence spaces and \( A = (a_{nk}) \) be an infinite matrix of real or complex numbers \( a_{nk} \), where \( n, k \in N_0 = \{0, 1, 2, ...\} \). Then, we say that \( A \) defines a matrix mapping from \( \lambda \) into \( \mu \), and we denote it by \( A : \lambda \to \mu \), if for every sequence \( x = (x_k) \in \lambda \), the sequence \( Ax = \{(Ax)_n\} \), the \( A \)-transform of \( x \), is in \( \mu \), where
\[
(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad n \in N_0, \tag{2.5}
\]

**Table 2.** Subdivisions of spectrum of a linear operator

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<td>is bounded</td>
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<td>I</td>
<td>( R(T - \lambda I) = X )</td>
<td>( \lambda \in \rho(T, X) )</td>
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<tr>
<td></td>
<td>( \lambda \in \sigma(T, X) )</td>
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<td></td>
<td>( \lambda \in \sigma_{ap}(T, X) )</td>
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<tr>
<td>II</td>
<td>( R(T - \lambda I) = X )</td>
<td>( \lambda \in \rho(T, X) )</td>
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<td></td>
<td>( \lambda \in \sigma_{co}(T, X) )</td>
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</tr>
<tr>
<td>III</td>
<td>( R(T - \lambda I) \neq X )</td>
<td>( \lambda \in \sigma_s(T, X) )</td>
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<tr>
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<td></td>
<td>( \lambda \in \sigma_{co}(T, X) )</td>
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</table>
By \((\lambda : \mu)\), we denote the class of all matrices such that \(A : \lambda \rightarrow \mu\). Thus, \(A \in (\lambda : \mu)\) if and only if the series on the right hand side of (2.5) converges for each \(n \in N\), and every \(x \in \lambda\), and we have \(Ax = \{ (Ax)_k \}_{k=1}^\infty \in \mu\) for all \(x \in \lambda\).

The lower triangular matrix \(B(r,s,t)\) is an infinite matrix of the form
\[
B(r,s,t) = \begin{bmatrix}
r & 0 & 0 & 0 & \cdots \\
s & r & 0 & 0 & \cdots \\
0 & t & s & r & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

We assume here and hereafter that \(s\) and \(t\) are complex parameters which do not simultaneously vanish.

The following results will be used in order to establish the results of this article.

**Lemma 2.2** [Wilansky (1984), Example 6B, Page 130]. The matrix \(A = (a_{nk})\) gives rise to a bounded linear operator \(T \in B(cs)\) from \(cs\) to itself if and only if
(i) \(\sup \sum_{k=1}^\infty |a_{nk} - a_{n,k-1}| < \infty\)
(ii) \(\sum_{k=1}^\infty a_{nk}\) is convergent for each \(k\).

**Lemma 2.3** [Golberg (1985), Page 59] \(T\) has a dense range if and only if \(T^*\) is one to one.

**Lemma 2.4** [Golberg (1985), Page 60] \(T\) has a bounded inverse if and only if \(T^*\) is onto.

3. Spectrum and fine spectrum of the operator \(B(r,s,t)\) on the sequence space \(cs\)

In this section, the fine spectrum of the operator \(B(r,s,t)\) on the sequence space \(cs\) has been examined.

Before giving the main theorem we should give the following remark. In this work, here and in follows, if \(z\) is a complex number then by \(\sqrt{z}\) we always mean the square root of with non-negative real part. If \(\text{Re}(\sqrt{z}) = 0\) then \(\sqrt{z}\) represents square root of \(z\) with \(\text{Im}(\sqrt{z}) \geq 0\). The same results are obtained if \(\sqrt{z}\) represents the square root.

**Theorem 3.1** \(B(r,s,t) : cs \rightarrow cs\) is a bounded linear operator and \(\|B(r,s,t)\|_{cs \rightarrow cs} \leq |r| + |s| + |t|\).

**Proof:** From Lemma 2.2, it is easy to show that \(B(r,s,t) : cs \rightarrow cs\) is a bounded linear operator. Now,

\[
B(r,s,t)(x) = \left| \sum_{k=0}^\infty a_{0k}x_k + \sum_{k=1}^\infty a_{1k}x_k + \sum_{k=2}^\infty a_{2k}x_k \right| \\
\leq |r| \sum_{k=0}^\infty x_k + |s| \sum_{k=1}^\infty x_k + |t| \sum_{k=2}^\infty x_k \\
\leq (|r| + |s| + |t|) \| x \|
\]

and hence, \(\|B(r,s,t)\|_{(cs \rightarrow cs)} \leq |r| + |s| + |t|\).

**Theorem 3.2** If \(s\) is a complex number such that \(\sqrt{s^2} = -s\), then \(\sigma (B(r,s,t), cs) = S\) where
\[
S = \left\{ \alpha \in C : \frac{2(r - \alpha)}{-s + \sqrt{s^2 - 4tr(\alpha - r)}} \leq 1 \right\}.
\]

**Proof:** We shall prove this theorem by showing that \((B(r,s,t) - \alpha I)^{-1}\) exists and is in \((cs : cs)\) for \(\alpha \in S\), and then show that the operator \((B(r,s,t) - \alpha I)^{-1}\) is not invertible for \(\alpha \in S\).

Without loss of any generality we assume that \(\sqrt{s^2} = -s\). Let \(\alpha \notin S\). Clearly \(\alpha \neq r\) and so \((B(r,s,t) - \alpha I)\) is a triangle, therefore \((B(r,s,t) - \alpha I)^{-1}\) exists. Let \(y = (y_n) \in cs\). Solving \((B(r,s,t) - \alpha I)x = y\) in terms of \(y\) we get

\[
x_0 = \frac{y_0}{r - \alpha} \\
x_1 = \frac{y_1}{r - \alpha} + \frac{-sy_0}{(r - \alpha)^2} \\
x_2 = \frac{y_2}{r - \alpha} + \frac{-sy_1}{(r - \alpha)^2} + \frac{s^2 - t(r - \alpha)}{(r - \alpha)^3}y_0 \\
\vdots
\]

Let us denote \(a_1 = \frac{1}{r - \alpha}, a_2 = \frac{-s}{(r - \alpha)^2}, a_3 = \frac{s^2 - t(r - \alpha)}{(r - \alpha)^3}\) etc.

Then, we have

\[
x_0 = a_0 y_0 \\
x_1 = a_1 y_1 + a_2 y_0 \\
x_2 = a_2 y_2 + a_3 y_1 + a_4 y_0 \\
\vdots
\]

\[
x_n = a_n y_n + a_{n+1} y_{n-1} + a_{n+2} y_{n-2} + \cdots + a_{n+\ell} y_0 = \sum_{k=0}^{\infty} a_{n+k} Y_k.
\]

That is
\[(B(r, s, t) - \alpha I)^{-1} = (a_n) = \begin{bmatrix} a_1 & 0 & 0 & 0 & \cdots \\ a_2 & a_1 & 0 & 0 & \cdots \\ a_3 & a_2 & a_1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \]

Also, from \((B(r, s, t) - \alpha I)x = y\), we have \(y_n = tx_n + sx_{n-1} + (r - \alpha)x_n\).

Using the recurrence relation \(x_n = \sum_{k=0}^{n} a_{n-k} y_k\), we get

\[
y_n = t \sum_{k=0}^{n} a_{n-k} y_k + s \sum_{k=0}^{n} a_{n-k} y_k + (r - \alpha) \sum_{k=0}^{n} a_{n-k} y_k = y_0 \left( ta_{n+1} + sa_{n+1} + (r - \alpha)a_{n+1} \right) + y_1 \left( ta_{n+1} + sa_{n+1} + (r - \alpha)a_{n+1} \right) + \cdots + y_n a_n (r - \alpha).
\]

This gives

\[
\begin{align*}
(ta_{n+1} + sa_{n+1} + (r - \alpha)a_{n+1}) &= 0 \\
(ta_{n+2} + sa_{n+2} + (r - \alpha)a_{n+2}) &= 0 \\
& \quad \vdots \\
(a_{n+1}, (r - \alpha) &= 1
\end{align*}
\]

This sequence can be obtained recursively by putting

\[
a_i = \frac{1}{r - \alpha}, \quad a_s = \frac{-s}{(r - \alpha)}, \quad ta_{n+1} + sa_{n+1} + (r - \alpha)a_{n+1} = 0, \quad n \geq 3.
\]

The characteristic equation of the recurrence relation is

\[(r - \alpha) \lambda^2 + s \lambda + t = 0.
\]

Then we have two cases:

Case 1: Let \(D = s^2 - 4t(r - \alpha) \neq 0\).

Then the roots of the characteristic equation are

\[
\lambda_1 = \frac{-s + \sqrt{D}}{2(r - \alpha)} \quad \text{and} \quad \lambda_2 = \frac{-s - \sqrt{D}}{2(r - \alpha)}.
\]

It is easy to show that \(a_s = \frac{\lambda_2 - \lambda_1}{\sqrt{D}}, \quad n \geq 1.

Since \(\alpha \notin S\), so \(|\lambda_1| < 1\) and therefore we have

\[
\left|1 + \frac{D}{s^2}\right| < \frac{2(r - \alpha)}{-s}.
\]

Since, \(1 - \sqrt{z} \leq 1 + \sqrt{z}\) for all \(z \in C\), so \(1 - \frac{D}{s^2} < \frac{2(r - \alpha)}{-s}\) and hence \(|\lambda_1| < 1\).

It is easy to show that for all \(m,
\[
\sum_{k=0}^{m} (a_{n-k} - a_{n-k-1}) \leq \sum_{k=0}^{m} |a_k| = \frac{1}{\sqrt{D}} \left( \sum_{k=0}^{m} |\lambda_1|^k + \sum_{k=0}^{m} |\lambda_2|^k \right)
\]

and hence, \(\sup_{k} \sum_{n=k}^{m} (a_{n-k} - a_{n-k-1}) < \infty\), as \(\frac{-s}{2(r - \alpha)} < 1\).
Since \[ \left| \frac{-s}{2(r-\alpha)} \right| < 1, \] so for all \( k \), the series \[ \sum_{s} a_{s} = a_{1} + a_{2} + a_{3} + \cdots \] is absolutely convergent and hence convergent. 

So, by Lemma 2.2, \( (B(r,s,t) - \alpha I)^{-1} \) is in \( (cs : cs) \).

This shows that \( \sigma(B(r,s,t), cs) \subseteq S \).

Next let \( \alpha \in S \). Then we have \[ \left| \frac{-s}{2(r-\alpha)} \right| \geq 1 \] from which we get \( \lim_{s \to \infty} a_{s} \neq 0 \) and so for all \( k \), the series \[ \sum_{s} a_{s} = a_{1} + a_{2} + a_{3} + \cdots \] is divergent. Therefore \( (B(r,s,t) - \alpha I)^{-1} \) is not in \( (cs : cs) \) and hence \( S \subseteq \sigma(B(r,s,t), cs) \).

Thus in each case we get \( \sigma(B(r,s,t), cs) = S \). This completes the proof.

**Remark:** If \( \sqrt{s^{2}} = s \), then we obtain the same sequence and 

\[ \sigma(B(r,s,t), cs) = \left\{ \alpha \in C : \frac{2(r-\alpha)}{s + \sqrt{s^{2} - 4t(r-\alpha)}} \leq 1 \right\} \]

**Theorem 3.3** The point spectrum of the operator \( B(r,s,t) \) over \( cs \) is given by \( \sigma_{p}(B(r,s,t), cs) = \emptyset \).

**Proof:** Let \( \alpha \) be an eigenvalue of the operator \( B(r,s,t) \). Then there exists \( x \neq \theta = (0,0,0,\ldots) \) in \( cs \) such that \( B(r,s,t)x = \alpha x \).

Then, we have 

\[ rx_{0} = \alpha x_{0} \]
\[ sx_{0} + rx_{1} = \alpha x_{1} \]
\[ tx_{0} + sx_{1} + rx_{2} = \alpha x_{2} \]
\[ tx_{1} + sx_{2} + rx_{3} = \alpha x_{3} \]
\[ \cdots \]
\[ tx_{n-2} + sx_{n-1} + rx_{n} = \alpha x_{n}, \quad n \geq 2 \]

If \( x_{k} \) is the first non-zero entry of the sequence \( (x_{k}) \), then \( \alpha = r \). Then from the relation \( tx_{k-1} + sx_{k} + rx_{k+1} = \alpha x_{k} \), we have \( x_{k} = 0 \), a contradiction.

Hence, \( \sigma_{p}(B(r,s,t), cs) = \emptyset \). This completes the proof. \( \square \)

If \( T : cs \to cs \) is a bounded linear operator represented by a matrix \( A \), then it is known that the adjoint operator \( T^*: cs^* \to cs^* \) is defined by the transpose \( A^* \) of the matrix \( A \).

It should be noted that the dual space \( cs^* \) of \( cs \) is isometrically isomorphic to the Banach space \( bv \) of all bounded variation sequences normed by \( \|x\|_{bv} = \sum_{s=1}^{\infty} |x_{s+1} - x_{s}| + \lim_{s \to \infty} |x_{s}| \).

**Theorem 3.4** The point spectrum of the operator \( B(r,s,t) \) over \( cs \) is given by \( \sigma_{p}(B(r,s,t), cs) \subseteq S \), where \( S = \left\{ \alpha \in C : \frac{2(r-\alpha)}{-s + \sqrt{s^{2} - 4t(r-\alpha)}} \leq 1 \right\} \).

**Proof:** Let \( \alpha \) be an eigenvalue of the operator \( B(r,s,t) \). Then there exists \( x \neq \theta = (0,0,0,\ldots) \) in \( bv \) such that \( B(r,s,t)x = \alpha x \). Then, we have 

\[ B(r,s,t)x = \alpha x \]
\[ \Rightarrow rx_{0} + sx_{1} + tx_{2} = \alpha x_{0} \]
\[ rx_{1} + sx_{2} + tx_{3} = \alpha x_{1} \]
\[ rx_{2} + sx_{3} + tx_{4} = \alpha x_{2} \]
\[ \cdots \]

If \( \alpha \in S \), then we may choose \( x_{0} \neq 0 \) and \( x = (x_{0}, 0, 0, \ldots) \) is an eigenvector corresponding to \( \alpha = r \).

Assume that \( \alpha \neq r \).

Then, we have

\[ x_{0} = \frac{-s}{t} x_{1} - \frac{r-\alpha}{t} x_{0} \]
\[ x_{1} = \frac{s^{2} - t(r-\alpha)}{t^{2}} x_{0} + \frac{s(r-\alpha)}{t^{2}} x_{0} \]
\[ \vdots \]
\[ x_{n} = \frac{a_{n}(r-\alpha)^{n}}{t^{n+1}} x_{1} - \frac{a_{n-1}(r-\alpha)^{n-1}}{t^{n+1}} x_{0}, \quad n \geq 2 \]

We now show that \( x_{n} = (x_{0})^{*}, \quad n \geq 2 \).

Since \( \lambda_{1} \) and \( \lambda_{2} \) are roots of the characteristic equation \( (r-\alpha)\lambda^{2} + s\lambda + t = 0 \), therefore

\[ \lambda_{1}\lambda_{2} = \frac{t}{r-\alpha}, \quad \lambda_{1} - \lambda_{2} = \frac{\sqrt{D}}{r-\alpha} \]
where \( \lambda_i = \frac{-s + \sqrt{D}}{2(r - \alpha)} \), \( \lambda_i = \frac{-s - \sqrt{D}}{2(r - \alpha)} \) and \( D = s^2 - 4t(r - \alpha) \neq 0 \).

Clearly \( x_i = \frac{1}{\lambda_i} \). Then we have

\[
x_{e} = \frac{a_{e}(r - \alpha)^{n}}{t - 1} x_{i} - \frac{a_{\alpha}(r - \alpha)^{n}}{t - 1} x_{s}, \quad n \geq 2
\]

\[
= \left( \frac{r}{t - 1} \right)^{n} x_{i} + \frac{a_{\alpha}(r - \alpha)^{n}}{t - 1} x_{e}, \quad n \geq 2
\]

\[
= \frac{r - \alpha}{(\lambda_i)^{n}} \left( \frac{1}{\lambda_i} \right) \frac{a_{\alpha}(r - \alpha)^{n}}{t - 1} x_{s}
\]

\[
= \left( \frac{\lambda_i}{\lambda_i - \lambda_2} \right) \frac{a_{\alpha}(r - \alpha)^{n}}{t - 1} x_{s}
\]

\[
= \left( \frac{x_{s}}{x_{i}} \right)^{n}
\]

If \( D = s^2 - 4t(r - \alpha) = 0 \) then also we may get the same result.

Now, \( \sum_{s=0}^{\infty} |x_{s}| - \lambda_i |x_{s}| \leq \sum_{s=0}^{\infty} |x_{s}| + \sum_{s=0}^{\infty} |x_{i}|^{n} < \infty \) as \( |x_{i}| < 1 \). Therefore \( x \in bv \).

Hence \( S_{i} \subseteq \sigma_{s}(B(r,s,t)^{+},cs^{*} \sim bv) \).

Next assume that \( \alpha \notin S_{i} \). Then \( \frac{2(r - \alpha)}{s + \sqrt{s^2 - 4t(r - \alpha)}} \geq 1 \) and so \( |\lambda_{i}| < 1 \). We must show that \( \alpha \notin \sigma_{s}(B(r,s,t),cs^{*} \sim bv) \).

Using \( x_{e} = \frac{a_{e}(r - \alpha)^{n}}{t - 1} x_{i} - \frac{a_{\alpha}(r - \alpha)^{n}}{t - 1} x_{s}, \quad n \geq 2 \), we get

\[
\frac{x_{e}}{x_{i}} = \left( \frac{r - \alpha}{t} \right) \frac{a_{e}}{a_{\alpha}} \left( \frac{-s + a_{e}}{a_{\alpha}} \right) x_{i}
\]

\[
= \left( \frac{x_{s}}{x_{i}} \right)^{n}
\]

We now consider three cases:

Case (i): \( |\lambda_{i}|, |\lambda_{2}| \leq 1 \)

In this case \( D = s^2 - 4t(r - \alpha) \neq 0 \) and

\[
\lim_{n \to \infty} \frac{a_{e}}{a_{\alpha}} = \lim_{n \to \infty} \frac{a_{e}}{a_{\alpha}} = \frac{\lambda_{i}+1}{\lambda_{2}+1} = \frac{\lambda_{i}+1}{\lambda_{2}+1}
\]

Now, if \( -x_{e} + \lambda_{2} x_{i} = 0 \), then we get \( x_{e} = \frac{x_{s}}{\lambda_{i}} \). Since \( |\lambda_{i}| < 1 \), therefore \( x_{s} \notin c \) and so \( x_{e} \notin bv \). Otherwise

\[
\lim_{n \to \infty} \frac{x_{e}}{x_{i}} = \frac{1}{|\lambda_{i}|} \frac{|\lambda_{2}|}{|\lambda_{2}|} = \frac{1}{|\lambda_{i}|} > 1.
\]

Case (ii): \( |\lambda_{i}|, |\lambda_{2}| < 1 \)

In this case \( D = s^2 - 4t(r - \alpha) = 0 \) and \( a_{e} = \frac{2n}{2s} \left( \frac{-s}{2t} \right)^{n} \),

\[
n \geq 1.
\]

Then

\[
\lim_{n \to \infty} \frac{x_{e}}{x_{i}} = \frac{1}{|\lambda_{i}|} \frac{|\lambda_{2}|}{|\lambda_{2}|} = \frac{1}{|\lambda_{i}|} > 1.
\]

Case (iii): \( |\lambda_{i}|, |\lambda_{2}| = 1 \)

In this case \( D = s^2 - 4t(r - \alpha) = 0 \) and we have \( \frac{-s}{2t} = 1 \).

Assume that \( \alpha \in \sigma_{s}(B(r,s,t)^{+},cs^{*} \sim bv) \). This implies that \( x \in bv \) and \( x \neq \theta \).

Again from \( x_{e} = \frac{a_{e}(r - \alpha)^{n}}{t - 1} x_{i} - \frac{a_{\alpha}(r - \alpha)^{n}}{t - 1} x_{s}, \quad n \geq 2 \), we get

\[
x_{e} = \left( \frac{-s}{2t} \right)^{n} \left( \frac{-s}{2t} \right) x_{s} + nx_{s}
\]

Now, \( x \in bv \) and so \( x \in c \). Therefore we must have \( x_{e} = x_{i} = 0 \). Which implies \( x = \theta \), a contradiction. So \( \alpha \notin \sigma_{s}(B(r,s,t),cs^{*} \sim bv) \).

In case (i) and case (ii) above, we have \( x_{s} \notin c \) and so \( x_{e} \notin bv \). In case (iii) by assuming \( \alpha \in \sigma_{s}(B(r,s,t),cs^{*} \sim bv) \) we get a contradiction.

This completes the proof.

**Theorem 3.5** The residual spectrum of the operator \( B(r,s,t) \) over \( cs \) is given by

\[
\sigma_{s}(B(r,s,t),cs) = S_{i}.
\]
Proof: Since, \( \sigma \left( B(r,s,t),cs \right) = \sigma \left( B(r,s,t)^{\dagger},cs^{\ast} \right) \setminus \sigma \left( B(r,t,s),cs \right) \), so we get the required result by using Theorem 3.3 and Theorem 3.4.

**Theorem 3.6** The continuous spectrum of the operator \( B(r,s,t) \) over are given by \( \sigma_{c}(B(r,s,t),cs) = S_{c} \), where

\[
S_{c} = \left\{ \alpha \in C : \frac{2(r-a)}{-s + \sqrt{s^{2} - 4tr(a-a)}} = 1 \right\}.
\]

Proof: Since, \( \sigma \left( B(r,s,t),cs \right) \) is the disjoint union of \( \sigma_{c}(B(r,s,t),cs), \sigma \left( B(r,s,t),cs \right) \) and \( \sigma \left( B(r,t,s),cs \right) \), therefore, by Theorem 3.3, Theorem 3.4 and Theorem 3.5, we get

\[
\sigma_{c}(B(r,s,t),cs) = \left\{ \alpha \in C : \frac{2(r-a)}{-s + \sqrt{s^{2} - 4tr(a-a)}} = 1 \right\} \square
\]

**Theorem 3.7** If \( \alpha = r \), then \( \alpha \in \text{III}_{c}(B(r,s,t),cs) \) if \( |t| < |s| \) and \( \alpha \in \text{III}_{c}(B(r,s,t),cs) \) if \( |t| \geq |s| \).

Proof: If \( \alpha = r \), the range of \( B(r,s,t) \) is not dense. So, from Table 2 and Theorem 3.5, we have \( \alpha \in \sigma_{c}(B(r,s,t),cs) \).

From Table 2, we get \( \sigma_{c}(B(r,s,t),cs) = \text{III}_{c}(B(r,s,t),cs) \cup \text{III}_{c}(B(r,t,s),cs) \).

Therefore, \( \alpha \in \text{III}_{c}(B(r,s,t),cs) \) or \( \alpha \in \text{III}_{c}(B(r,t,s),cs) \).

Also for \( \alpha = r \), \( B(r,s,t) = -1 \) is \( B(0,s,t) \).

A left inverse of \( B(0,s,t) \) is

\[
(B(0,s,t))^{-1} = \begin{bmatrix}
0 & \frac{1}{s} & 0 & 0 & \cdots \\
0 & \frac{(-t)}{s^{2}} & \frac{1}{s} & 0 & \cdots \\
0 & \frac{(-t)^{2}}{s^{3}} & \frac{(-t)}{s^{2}} & \frac{1}{s} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

In other words \( (B(0,s,t))^{-1} = (b_{n},) \), where

\[
b_{n} = \begin{cases}
\frac{(-t)^{n+1}}{s^{n+2}} & \text{if } 1 \leq k \leq n+2 \\
0 & \text{if } k = 1 \text{ or } k \geq n+2
\end{cases}
\]

Now for each \( m \), we get

\[
\sum_{n=1}^{m} \left| \sum_{k=1}^{n} (b_{a_{k}} - b_{a_{k+1}}) \right| \leq \frac{1}{|s|} \frac{|t|}{|t|} + \frac{|t|}{|t|} + \cdots + \frac{|t|}{|t|} \text{ and so}
\]

Also for each \( k \), \( \sum_{n=1}^{m} b_{n} = \frac{1}{s} + \frac{-t}{s} + \frac{(-t)^{2}}{s} + \cdots \) is convergent if and only if \( |t| < |s| \).

Therefore, the matrix \( (B(0,s,t))^{-1} \) is in \( (cs : cs) \) if \( |t| < |s| \) and not in \( (cs : cs) \) if \( |t| \geq |s| \).

This completes the theorem.

**Theorem 3.8** If \( \alpha \neq r \) and \( \alpha \in \sigma_{c}(B(r,s,t),cs) \), then \( \alpha \in \text{III}_{c}(B(r,s,t),cs) \).

Proof: Since, \( \alpha \in \sigma_{c}(B(r,s,t),cs) \), therefore, from Table 2, we have \( \alpha \in \text{III}_{c}(B(r,s,t),cs) \) or \( \alpha \in \text{III}_{c}(B(r,t,s),cs) \).

Now, \( \alpha \in \sigma_{c}(B(r,s,t),cs) \) implies that \( \frac{2(r-a)}{-s + \sqrt{s^{2} - 4tr(a-a)}} < 1 \) and so \( |t| > 1 \)

Therefore, the series (3.1) in Theorem 3.2 is not convergent and hence, the operator \( B(r,s,t) \) has no bounded inverse. Therefore, \( \alpha \in \text{III}_{c}(B(r,s,t),cs) \).

**Theorem 3.9** The approximate point spectrum of the operator \( B(r,s,t) \) over is given by \( \sigma_{a}(B(r,s,t),cs) = \begin{cases}
S \setminus \{ \alpha \} & \text{if } |t| < |s| \\
S & \text{if } |t| \geq |s|
\end{cases} \)

Proof: From Table 2, we have \( \sigma_{a}(B(r,s,t),cs) = \sigma(B(r,s,t),cs) \setminus \text{III}_{c}(B(r,t,s),cs) \).

Using Theorem 3.2 and Theorem 3.7, we get the required result.

**Theorem 3.10** The compression spectrum of the operator \( B(r,s,t) \) over is given by

\[
\sigma_{c}(B(r,s,t),cs) = S_{c}.
\]

Proof: By proposition 2.1 (e), we get \( \left\lfloor B(r,s,t)^{\dagger},cs^{\ast} \right\rfloor = \sigma_{a}(B(r,s,t),cs) \).

Using Theorem 3.4, we get the required result.

**Theorem 3.11** The defect spectrum of the operator \( B(r,s,t) \) over is given by

\[
\sigma_{d}(B(r,s,t),cs) = S_{d}.
\]
Proof: From Table 2, we have $\sigma_{\ell}(B(r,s,t),cs) = \sigma(B(r,s,t),cs) \cup \sigma(B(r,s,t),cs)$. Also, $\sigma_{\ell}(B(r,s,t),cs) = I_{\sigma}(B(r,s,t),cs) \cup I_{\sigma}(B(r,s,t),cs) \cup I_{\sigma}(B(r,s,t),cs)$. By Theorem 3.3, we have $\sigma_{\ell}(B(r,s,t),cs) = \emptyset$ and so $I_{\sigma}(B(r,s,t),cs) = \emptyset$.

Hence $\sigma_{\ell}(B(r,s,t),cs) = S$.

**COROLLARY 3.13** The following statements hold:

(i) $\sigma_{\ell}(B(r,s,t),cs^\ast \equiv bv) = S$.

(ii) $\sigma_{\ell}(B(r,s,t),cs^\ast \equiv bv) = \left\{ S \setminus \{r\}, \text{ if } |r| < |s| \right\} \cup \left\{ S, \text{ if } |r| \geq |s| \right\}$.

**Proof:** Using Proposition 2.1 (c) and (d), we get

$\sigma_{\ell}(B(r,s,t),cs^\ast \equiv bv) = \sigma_{\ell}(B(r,s,t),cs)$ and

$\sigma_{\ell}(B(r,s,t),cs^\ast \equiv bv) = \sigma_{\ell}(B(r,s,t),cs)$.

Using Theorem 3.9 and Theorem 3.11, we get the required results.

References

Akhmedov, A.M. and Başar, F. 2008. The fine spectra of the Cesàro operator $C_\ell$ over the sequence space $bv_{p}$, $(1 \leq p < \infty)$. Mathematics Journal of Okayama University. 50, 135-147.

Akhmedov, A.M. and El-Shabrawy, S. R. 2011. On the fine spectrum of the operator $\Delta_\ell$ over the sequence space $c_\ell$, Computers and Mathematics with Applications. 61(10), 2994-3002.


Bilgiç, H. and Furkan, H. 2007. On the fine spectrum of operator $B(r,s,t)$ over the sequence spaces $\ell_1$ and $c$, Mathematics and Computer Modelling. 45, 883-891.

Bilgiç, H. and Furkan, H. 2008. On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces $\ell_1$ and $bv_{p}$, $(1 < p < \infty)$, Nonlinear Analysis. 68, 499-506.

Bilgiç, H., Furkan, H., and Kayaduman, K. 2006. On the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces $\ell_1$ and $bv_{p}$, Hokkaido Mathematical Journal. 35, 893-904.

Furkan, H., Bilgiç, H., and Altay, B. 2007. On the fine spectrum of operator $B(r,s,t)$ over $c_\ell$ and $c$, Computers and Mathematics with Applications. 53, 989-998.

Furkan, H., Bilgiç, H., and Başar, F. 2010. On the fine spectrum of operator $B(r,s,t)$ over the sequence spaces $\ell_1$ and $bv_{p}$, $(1 < p < \infty)$, Computers and Mathematics with Applications. 60, 2141-2152.


Kayaduman, K. and Furkan, H. 2006. The fine spectra of the difference operator $\Delta$ over the sequence spaces $\ell_1$ and $bv_{p}$, International Mathematics Forum, I(24), 1153-1160.


Panigrahi, B.L. and Srivastava, P.D. 2011. Spectrum and the fine spectrum of the generalised second order difference operator $\Delta_{w_1}$ on sequence space $c_{w_0}$, Thai Journal of Mathematics. 12(1), 57-74.

Panigrahi, B.L. and Srivastava, P.D. 2012. Spectrum and fine spectrum of the generalised second order forward difference operator $\Delta_{w_0}^{\ell_1}$ on the sequence space $\ell_1$, Bulletin of the Malaysian Mathematical Sciences Society. Article ID 161209.

Srivastava, P.D. and Kumar, S. 2010. Fine spectrum of the generalised difference operator $\Delta_{w_1}$ on the sequence space $\ell_1$, Thai Journal of Mathematics. 8(2), 221-233.

Tripathy, B.C. and Das, R. 2014. Spectra of the Rhaly operator on the sequence space $bv_{p} \cap \ell_1$, Boletim da Sociedade
Tripathy, B.C. and Das, R. 2015. Spectrum and fine spectrum of the upper triangular matrix $B(r, 0, s)$ over the sequence space, Applied Mathematics and Information Sciences. 9(4), 2139-2145. doi.org/10.12785/amis/090453

Tripathy, B.C. and Paul, A. 2013. Spectra of the operator $B(f, g)$ on the vector valued sequence space $c_0(\lambda)$. Boletim da Sociedade Paranaense de Matemática. 31(1), 105-111.

Tripathy, B.C. and Paul, A. 2013. The Spectrum of the operator $D(r,0,0,s)$ over the sequence space $c_0$ and $c$. Kyungpook Mathematical Journal. 53(2), 247-256.