More on generalized $b$-closed sets in double fuzzy topological spaces

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Abstract

The purpose of this paper is to introduce and study a new class of sets called $(r, s)$-generalized fuzzy $b$-regular weakly closed sets which lies between the new class of $(r, s)$-fuzzy regular weakly closed sets and $(r, s)$-generalized fuzzy $b$-closed sets in double fuzzy topological spaces. Several fundamental properties are introduced and discussed. Furthermore, the relationships between the new concepts are introduced and established with some interesting counter examples.

Keywords: double fuzzy topology, $(r, s)$-fuzzy regular weakly closed sets, $(r, s)$-generalized fuzzy $b$-regular weakly closed sets

1. Introduction

After the development of Zadeh’s theory of fuzzy sets Zadeh (1965), the idea of intuitionistic fuzzy set was first introduced by Atanassov (1993), then Çoker (1997) introduced the notion of intuitionistic fuzzy topological space. After that Samanta and Mondal (2002), introduced the notion of intuitionistic gradation of openness of fuzzy sets and gave the definition of intuitionistic fuzzy topological space as a generalization of smooth topology and the topology of intuitionistic fuzzy sets. Working under the name “intuitionistic” did not continue and ended in 2005 by Gutierrez Garcia and Rodabaugh (2005) when they proved that this term is unsuitable in mathematics and applications and they concluded that they work under the name “double”.

In 2009, Omari and Noorani (2009) introduced the class of generalized $b$-closed sets (briefly, $gb$-closed) in topological spaces. As a generalization of this work, we will apply the notions of $(r, s)$-generalized $wp$-closed sets (see Mohammed et al. (2013, 2014)) to introduce and study the $(r, s)$-generalized fuzzy $b$-regular weakly closed sets in double fuzzy topological spaces. The new notion lies between the class of $(r, s)$-fuzzy regular weakly closed sets and the class of $(r, s)$-generalized fuzzy $b$-closed sets. Finally, some inter-relations between the new concepts are introduced and established with some interesting counter examples.

2. Preliminaries

Throughout this paper, $X$ will be a non-empty set, $I$ is the closed unit interval $[0,1]$, $I_0 = (0,1]$ and $I_1 = [0,1)$. A fuzzy set $\lambda$ is quasi-coincident with a fuzzy set $\mu$ denoted by $\lambda \sim \mu$ if and only if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$ and they are not quasi-coincident otherwise which denoted by $\lambda \nsim \mu$ (see Pao-Ming and Ying-Ming (1980)). The family of all fuzzy sets on $X$ is denoted by $I^X$. By $\emptyset$ and $\mathbb{1}$, we denote the smallest and the largest fuzzy sets on $X$. For a fuzzy set $\lambda \in I^X$, $\mathbb{1} - \lambda$ denotes its complement. All other notations are standard notations of fuzzy set theory.

Now, we recall the following definitions which are useful in the sequel.
Definition 2.1 (see Samanta and Mondal (2002)) A double fuzzy topology \((\tau, \tau^*)\) on \(X\) is a pair of maps \(\tau, \tau^*: I^X \rightarrow I\), which satisfies the following properties:

1. \(\tau(\lambda) \leq 1 - \tau^*(\lambda)\) for each \(\lambda \in I^X\).
2. \(\tau(\lambda \land \lambda_2) \geq \tau(\lambda) \land \tau(\lambda_2)\) and \(\tau^*(\lambda \land \lambda_2) \leq \tau^*(\lambda) \lor \tau^*(\lambda_2)\) for each \(\lambda, \lambda_2 \in I^X\).
3. \(\tau(\vee_{\alpha r} \lambda_2) \geq \tau^*(\lambda_2)\) and \(\tau^*(\vee_{\alpha r} \lambda_2) \leq \tau^*(\lambda_2)\) for each \(\alpha, \lambda_2 \in I^X\).

The triplet \((X, \tau, \tau^*)\) is called a double fuzzy topological space (dfts, for short). A fuzzy set \(\lambda\) is called an \((r, s)\)-fuzzy open \((r, s)\)-frso, for short) if \(\tau(\lambda) \geq r\) and \(\tau^*(\lambda) \leq s\). A fuzzy set \(\lambda\) is called an \((r, s)\)-fuzzy closed \((r, s)\)-fsc, for short) if \(1 - \lambda\) is an \((r, s)\)-frso set.

Theorem 2.1 (see Lee and Im (2001)) Let \((X, \tau, \tau^*)\) be a dfts. Then double fuzzy closure operator and double fuzzy interior operator of \(\lambda \in I^X\) are defined by

\[ C_{\tau, \tau^*}(\lambda, r, s) = \lambda \setminus \{ \mu \in I^X | \mu \leq \lambda, \tau(\mu) \geq r, \tau^*(\mu) \leq 1 - \mu \leq s \} \]

\[ I_{\tau, \tau^*}(\lambda, r, s) = \lambda \cap \{ \mu \in I^X | \mu \leq \lambda, \tau(\mu) \geq r, \tau^*(\mu) \leq 1 - \mu \leq s \} \] Where \(r \in I_0\) and \(s \in I_1\) such that \(r + s \leq 1\).

Definition 2.2 Let \((X, \tau, \tau^*)\) be a dfts. For each \(\lambda \in I^X\), \(r \in I_0\) and \(s \in I_1\). A fuzzy set \(\lambda\) is called:

1. An \((r, s)\)-regular fuzzy open (see Ramadan et al. (2005)) (briefly, \((r, s)\)-frso) if \(\lambda = (I_{\tau, \tau^*} (C_{\tau, \tau^*}(\lambda, r, s), r, s), s)\). \(\lambda\) is called an \((r, s)\)-regular fuzzy closed (briefly, \((r, s)\)-frsc) if \(1 - \lambda\) is an \((r, s)\)-frso set.
2. An \((r, s)\)-fuzzy semi open (see Kim and Abbas (2004)) (briefly, \((r, s)\)-fsso) if \(\lambda \subseteq C_{\tau, \tau^*} (I_{\tau, \tau^*} (C_{\tau, \tau^*}(\lambda, r, s), r, s), s)\). \(\lambda\) is called an \((r, s)\)-fuzzy semi closed (briefly, \((r, s)\)-fscc) if \(1 - \lambda\) is an \((r, s)\)-fsso set.
3. An \((r, s)\)-regular generalized fuzzy closed (see Gharieb (2011)) (briefly, \((r, s)\)-rgfc) if \(C_{\tau, \tau^*}(\lambda, r, s) \subseteq \lambda, \tau(\mu) \geq r, \tau^*(\mu) \leq 1 - \mu\) if \(\lambda \subseteq \lambda, \mu \leq \mu, \mu\) is an \((r, s)\)-frso.
4. An \((r, s)\)-fuzzy regular semiopen (see El-Saady and Gharieb (2012)) (briefly, \((r, s)\)-frso) if there exists an \((r, s)\)-fuzzy regular open \(\lambda\) such that \(\lambda \subseteq C_{\tau, \tau^*} (\lambda, r, s)\).
5. An \((r, s)\)-generalized fuzzy closed (see Abbas and El-Sanousy (2012)) (briefly, \((r, s)\)-gcc) if \(\lambda \subseteq C_{\tau, \tau^*}(\lambda, r, s) \subseteq \lambda, \mu \leq \mu, \tau(\mu) \geq r, \tau^*(\mu) \leq 1 - \mu\) is called an \((r, s)\)-generalized fuzzy open \(\lambda\) such that \(\lambda \subseteq \lambda, \mu \leq \mu, \mu\) is an \((r, s)\)-frso.
6. An \((r, s)\)-fuzzy b-closed (see Mohammed et al. (2015)) (briefly, \((r, s)\)-fbc) if \(\lambda \geq (I_{\tau, \tau^*}(C_{\tau, \tau^*}(\lambda, r, s), r, s)) \land (C_{\tau, \tau^*}(I_{\tau, \tau^*}(\lambda, r, s), r, s))\). \(\lambda\) is called an \((r, s)\)-fuzzy b-open \(\lambda\) such that \(\lambda \subseteq \lambda, \mu \leq \mu, \mu\) is an \((r, s)\)-frso.
7. An \((r, s)\)-generalized fuzzy b-closed (see Mohammed et al. (2015)) (briefly, \((r, s)\)-gbc) if \(bC_{\tau, \tau^*}(\lambda, r, s) \subseteq \lambda, \lambda \subseteq \lambda, \mu \leq \mu, \mu\) is an \((r, s)\)-frso.
Then $\alpha$ is an $(\frac{1}{2}, \frac{1}{2})$-gfbrwc set.

**Theorem 3.2** Let $(X, r, e^*)$ be a dfts, $\lambda \in I^X$, $r \in I_0$ and $s \in I_1$. If $\lambda$ is an $(r, s)$-gfbrwc set, then

1. $b_{C_{r,s}^e}((\lambda, r, s) - \lambda)$ does not contain any non-zero $(r, s)$-frso sets.

2. $\lambda$ is an $(r, s)$-frso iff $b_{C_{r,s}^e}((\lambda, r, s) - \lambda)$ is $(r, s)$-frfc.

3. $\mu$ is an $(r, s)$-gfbrwc set for each set $\mu \in I^X$ such that $\lambda \subseteq \mu \subseteq b_{C_{r,s}^e}((\lambda, r, s))$.

4. For each $(r, s)$-frso set $\mu \in I^X$ such that $\mu \subseteq \lambda$, $\mu$ is an $(r, s)$-gfbrwc relative to $\lambda$, $\mu$ is an $(r, s)$-gfbrwc in $X$.

5. $b_{C_{r,s}^e}((\lambda, r, s) - \lambda)$ does not contain any non-zero $(r, s)$-frso sets.

**Proof.** (1) Suppose that $\mu$ is a non-zero $(r, s)$-frso set of $I^X$ such that $\mu \subseteq b_{C_{r,s}^e}((\lambda, r, s) - \lambda)$ whenever $\lambda \in I^X$ is an $(r, s)$-frso set, $r \in I_0$, and $s \in I_1$. Hence, $\mu \subseteq 1 - \lambda$ or $\lambda \subseteq 1 - \mu$. Since $1 - \mu$ is an $(r, s)$-frso set. But, $\lambda$ is an $(r, s)$-gfbrwc set, hence

$$\lambda \subseteq (1 - \mu) \Rightarrow b_{C_{r,s}^e}((\lambda, r, s) - (1 - \mu)) \Rightarrow (1 - b_{C_{r,s}^e}((\lambda, r, s))) \land (b_{C_{r,s}^e}((\lambda, r, s) - \lambda) = 0$$

and hence $\mu = 0$ which is a contradiction. Then $b_{C_{r,s}^e}((\lambda, r, s) - \lambda)$ does not contain any non-zero $(r, s)$-frso sets.

(2) Suppose $\lambda$ is an $(r, s)$-gfbrwc set. So, for each $r \in I_0$ and $s \in I_1$ if $\lambda$ is an $(r, s)$-frso set then, $b_{C_{r,s}^e}((\lambda, r, s) - \lambda) = 0$ which is an $(r, s)$-frfc set.

Conversely, suppose $b_{C_{r,s}^e}((\lambda, r, s) - \lambda$ is an $(r, s)$-frso set. Then by (1), $b_{C_{r,s}^e}((\lambda, r, s) - \lambda$ does not contain any non-zero $(r, s)$-frso set. But $b_{C_{r,s}^e}((\lambda, r, s) - \lambda$ is an $(r, s)$-frfc set, then

$$b_{C_{r,s}^e}((\lambda, r, s) - \lambda = 0 \Rightarrow \lambda = b_{C_{r,s}^e}((\lambda, r, s).$$

So, $\lambda$ is an $(r, s)$-frfc set.

(3) Suppose $\alpha$ is an $(r, s)$-frso set such that $\mu \subseteq \alpha$ and let $\lambda$ be an $(r, s)$-gfbrwc set such that $\lambda \subseteq \alpha$, $r \in I_0$, and $s \in I_1$. Then

$$b_{C_{r,s}^e}((\lambda, r, s) \subseteq \alpha,$$

hence,

$$b_{C_{r,s}^e}((\mu, r, s) \subseteq b_{C_{r,s}^e}((b_{C_{r,s}^e}((\lambda, r, s), r, s) = b_{C_{r,s}^e}((\lambda, r, s) \subseteq \mu).$$

Therefore $\mu$ is an $(r, s)$-gfbrwc set.

(4) Suppose that $\mu$ is an $(r, s)$-gfbrwc and $\nu$ is an $(r, s)$-frso in $I^X$ such that $\mu \subseteq \nu$, $r \in I_0$, $s \in I_1$. But $\lambda \leq 1$, therefore $\mu \leq \lambda$ and $\mu \leq \nu$. So

$$\mu \leq \lambda \land \nu.$$

But $\mu$ is an $(r, s)$-gfbrwc relative to $\lambda$,

$$\lambda \land b_{C_{r,s}^e}((\mu, r, s) \subseteq \lambda \land \nu \Rightarrow \lambda \land b_{C_{r,s}^e}((\mu, r, s) \subseteq \nu.$$
(2) Since every an (r, s)-fc set is an (r, s)-gfbwr and from (1) we get the proof.

**Proposition 3.2** Let \((X, \tau, \tau^*)\) be dfts’s. For each \(\lambda\) and \(\mu \in \mathbb{I}\), \(r \in I_\mu\) and \(s \in I_\lambda\).

1. If \(\tau(\lambda) \geq r\) and \(\tau^*(\lambda) \leq s\), such that \(\lambda\) is an (r, s)-gfb set. Then \(\lambda\) is an (r, s)-gbwr-closed set.

2. If \(\lambda\) is both an (r, s)-frs and an (r, s)-gbwr set, then \(\lambda\) is an (r, s)-frs-clopen.

3. If \(\lambda\) is both an (r, s)-frs and an (r, s)-rgfc, then \(\lambda\) is an (r, s)-rgfc set.

4. If \(\lambda\) is both an (r, s)-frs and an (r, s)-gfbwr set, then \(\lambda\) is an (r, s)-frs-set.

5. If \(\lambda\) is both an (r, s)-frs and an (r, s)-gfbwr set, such that \(\tau(1 - \mu) \geq r\) and \(\tau^*(1 - \mu) \leq s\). Then \(\lambda \land \mu\) is an (r, s)-gfbwr set.

**Proof.**

1. Suppose that \(\lambda \leq \mu\) and \(\mu\) is an (r, s)-frs in \(I^X\) such that \(r \in I_\mu\) and \(s \in I_\lambda\). Since \(\tau(\lambda) \geq r\) and \(\tau^*(\lambda) \leq s\) such that \(\lambda\) is an (r, s)-gfb set and \(\lambda \leq \mu\), then

\[
bC_{\tau, \tau^*}(\lambda, r, s) \leq \lambda \Rightarrow bC_{\tau, \tau^*}(\lambda, r, s) \leq \mu.
\]

Hence, \(\lambda\) is an (r, s)-gfbwr set.

2. Suppose that \(\lambda\) is an (r, s)-frs and an (r, s)-gfbwr set, \(r \in I_\mu\) and \(s \in I_\lambda\). Since every (r, s)-frs set is (r, s)-frs set and \(\lambda \leq \lambda\), \(bC_{\tau, \tau^*}(\lambda, r, s) \leq \lambda\). Also, \(\mu \leq bC_{\tau, \tau^*}(\lambda, r, s)\). Therefore, \(\lambda = bC_{\tau, \tau^*}(\lambda, r, s)\), that is \(\lambda\) is (r, s)-frs set. But, \(\lambda\) is an (r, s)-frs set, hence, \(\lambda\) is (r, s)-frs set. Therefore, \(\lambda\) is an (r, s)-frs-clopen set.

3. Suppose that \(\mu\) is an (r, s)-frs in \(I^X\) such that \(\lambda \leq \mu\), \(r \in I_\mu\) and \(s \in I_\lambda\). But, \(\lambda\) is an (r, s)-frs and an (r, s)-rgfc. Then by (2), \(bC_{\tau, \tau^*}(\lambda, r, s) \leq \lambda\). Hence, \(bC_{\tau, \tau^*}(\lambda, r, s) \leq \mu\). So, \(\lambda\) is an (r, s)-gfbwr set.

4. Obvious.

5. Suppose that \(\lambda\) is an (r, s)-frs and an (r, s)-gfbwr set of \(I^X\), \(r \in I_\mu\) and \(s \in I_\lambda\). Then by (4), \(\lambda\) is an (r, s)-frs-clopen. But, \(\tau(1 - \mu) \geq r\) and \(\tau^*(1 - \mu) \leq s\), so \(\lambda \land \mu\) is an (r, s)-gfb set. Therefore, \(\lambda \land \mu\) is an (r, s)-gfbwr set.

### 4. Interrelations

The following implication illustrates the relationships between different fuzzy sets:

\[
(r, s) - \text{fc} \rightarrow (r, s) - \text{frwc} \rightarrow (r, s) - \text{gfbwr} \rightarrow (r, s) - \text{gbc}
\]

None of these implications is reversible where \(A \rightarrow B\) represents \(A\) implies \(B\), as shown by the following examples.

#### Example 4.1

1. Let \(X = \{a, b, c\}\) and let \(\mu_1, \mu_2\) and \(\mu_3\) are fuzzy sets defined as follows:

\[
\mu_1(a) = 0.7, \mu_1(b) = 0.7, \mu_1(c) = 0.7,
\]

\[
\mu_2(a) = 0.6, \mu_2(b) = 0.7, \mu_2(c) = 0.7,
\]

\[
\mu_3(a) = 0.4, \mu_3(b) = 0.6, \mu_3(c) = 0.7.
\]

Defined \((r, \tau^*)\) on \(X\) as follows:

\[
\begin{cases}
1, & \text{if } \lambda \in [0.4, 0.6], \\
0.3, & \text{if } \lambda = \mu, \\
0.6, & \text{if } \lambda = \mu, \\
0, & \text{otherwise},
\end{cases}
\]

Then \(\mu_1\) is an \((0.3, 0.6)-frwc\) set, but not an \((0.3, 0.6)-frw\) set.

2. Let \(X = \{a, b, c\}\) and let \(\mu_1, \mu_2\) and \(\mu_3\) are fuzzy sets defined as follows:

\[
\begin{cases}
\mu_1(a) = 0.3, \mu_1(b) = 0.3, \mu_1(c) = 0.3, \\
\mu_2(a) = 0.3, \mu_2(b) = 0.3, \mu_2(c) = 0.3, \\
\mu_3(a) = 0.4, \mu_3(b) = 0.3, \mu_3(c) = 0.3.
\end{cases}
\]

Defined \((r, \tau^*)\) on \(X\) as follows:

\[
\begin{cases}
1, & \text{if } \lambda \in [0.4, 0.6], \\
0.6, & \text{if } \lambda = \mu, \\
0, & \text{otherwise},
\end{cases}
\]

Then \(\mu_2\) is an \((0.6, 0.3)-gfb\) set, but not an \((0.6, 0.3)-gbc\) set.

### References


