Original Article

On \((e_e \lor q_k)\)-intuitionistic (fuzzy ideals, fuzzy soft ideals) of subtraction algebras

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Received: 27 February 2015; Accepted: 29 April 2015

Abstract

The intent of this article is to study the concept of an \((e_e \lor q_k)\)-intuitionistic fuzzy ideal and \((e_e \lor q_k)\)-intuitionistic fuzzy soft ideal of subtraction algebras and to introduce some related properties.

2010 AMS Classification: 06F35, 03G25, 08A72.

Keywords: Subtraction algebras, \((e_e \lor q_k)\)-intuitionistic fuzzy (soft) subalgebras, \((e_e \lor q_k)\)-intuitionistic fuzzy (soft) ideals.

1. Introduction

Schein (1992) cogitated the system of the form \((X; \circ, \setminus)\), where \(X\) is the set of functions closed under the composition \(\circ\) of functions (and hence \((X, \circ)\) is a function semigroup) and the set theoretical subtraction \(\setminus\) (and hence \((X, \setminus)\)) is a subtraction algebra in the sense of Abbot (1969). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. He suggested a problem concerning the structure of multiplication in a subtraction semigroup. It was explained by Zelinka (1995), and he had solved the problem for subtraction algebras of a special type known as the "atomic subtraction algebras". The notion of ideals in subtraction algebras was introduced by Jun et al. (2005). For detailed study of subtraction algebras see (Ceven and Ozturk, 2009 and Jun et al., 2007).

The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh (1965). Since then the notion of fuzzy sets is actively applicable in different algebraic structures. The fuzzification of ideals in subtraction algebras were discussed in Lee and Park (2007). Atanassov (1986) introduced the idea of intuitionistic fuzzy set, which is more general one as compared to a fuzzy set.

Bhakat and Das (1996) introduced a new type of fuzzy subgroups, that is, the \((e_e \lor q)\)-fuzzy subgroups. Jun et al. (2011) introduced the notion of \((e_e \lor q)\)-fuzzy subgroup. In fact, the \((e_e \lor q)\)-fuzzy subgroup is an important generalization of Rosenfeld’s fuzzy subgroup. Shabir et al. (2010) characterized semigroups by \((e_e \lor q)\)-fuzzy ideals, also see Shabir and Mahmood (2011, 2013).

Molodtsov (1999) introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that are free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, work on the soft set theory is progressing rapidly. Maji et al. (2002)
described the application of soft set theory to a decision-making problem. Maji et al. (2003) also studied several operations on the theory of soft sets. Maji et al. (2001b, 2004) studied intuitionistic fuzzy soft sets.

The notion of fuzzy soft sets, as a generalization of the standard soft sets, was introduced by Maji (2001a), and an application of fuzzy soft sets in a decision-making problem was presented. Ahmad et al. (2009) have introduced arbitrary fuzzy soft union and fuzzy soft intersection. Aygunoglu et al. (2009) introduced the notion of fuzzy soft group and studied its properties. Yaqoob et al. (2013) studied the properties of intuitionistic fuzzy soft groups in terms of intuitionistic double t-norm. Jun et al. (2010) introduced the notion of fuzzy soft BCK/BCI-algebras and (closed) fuzzy soft ideals, and then derived their basic properties. Williams and Saeid (2009) have introduced arbitrary fuzzy soft BCK/BCI-algebras and (closed) fuzzy soft ideals, and then derived their basic properties. Jun et al. (2009) introduced the notion of fuzzy soft group and studied fuzzy soft union and fuzzy soft intersection. Aygunoglu et al. (2010) introduced the notion of intuitionistic fuzzy soft groups in terms of intuitionistic fuzzy soft groups (ideals) in detail. Liu and Xin (2013) studied the idea of generalized fuzzy soft groups and fuzzy normal soft groups. Recently, Yang (2011) have studied fuzzy soft semigroups (ideals) in detail. Liu and Xin (2013) studied the idea of generalized fuzzy soft groups and fuzzy normal soft groups.

In this article, we study the concept of (, )-intuitionistic fuzzy (soft) ideals of subtraction algebras. Here we consider some basic properties of (, )-intuitionistic fuzzy (soft) ideals of subtraction algebras.

2. Preliminaries

In this section we recall some of the basic concepts of subtraction algebra which will be very helpful in further study of the paper. Throughout the paper denotes the subtraction algebra unless otherwise specified.

Definition 2.1 (Jun et al., 2005) A non-empty set together with a binary operation “−” is said to be a subtraction algebra if it satisfies the following:

\[ S_1 \ x - (y - x) = x, \]
\[ S_2 \ x - (x - y) = y - (y - x), \]
\[ S_3 \ (x - y) - z = (x - z) - y, \text{ for all } x, y, z \in X. \]

The last identity permits us to omit parentheses in expression of the form \((x - y) - z\). The subtraction determines an order relation on \(X: a \leq b \iff a - b = 0\), where \(0 = a - a\) is an element that does not depends upon the choice of \(a \in X\). The ordered set \((X, \leq)\) is a semi-Boolean algebra in the sense of Abbot (1969), that is, it is a meet semi lattice with zero, in which every interval \([0, a]\) is a Boolean algebra with respect to the induced order. Here \(a \land b = a - (a - b)\); the complement of an element \(b \in [0, a]\) is \(a - b\) and is denoted by \(b^c\); and if \(b, c \in [0, a]\); then \(b \lor c = \left(b \lor c\right) = ((a - b) \land (a - c)) - a - (a - b) - ((a - b) - (a - c)).\)

In a subtraction algebra, the following are true see (Jun et al., 2005):

\[ (a1) \ x - y = y - x, \]
\[ (a2) \ x - 0 = x \text{ and } 0 - x = 0, \]
\[ (a3) \ (x - y) - x = 0, \]
\[ (a4) \ x - (x - y) \leq y, \]
\[ (a5) \ x - (x - y) = y - x = x - y, \]
\[ (a6) \ x - (x - (y - x)) = x - y, \]
\[ (a7) \ x - (y - z) = (x - z) - (y - z). \]

Definition 2.2 (Jun et al., 2005) A non-empty subset \(A\) of a subtraction algebra \(X\) is called an ideal of \(X\), denoted by \(A \triangleleft X\): if it satisfies:

\[ (b1) \ a - x \in A \text{ for all } a \in A \text{ and } x \in X, \]
\[ (b2) \text{ for all } a, b \in A \text{ whenever } a \lor b \text{ exists in } X \text{ then } a \lor b \in A. \]

Proposition 2.3 (Jun et al., 2005) A non-empty subset \(A\) of a subtraction algebra \(X\) is called an ideal of \(X\), if and only if it satisfies:

\[ (b3) \ 0 \in A, \]
\[ (b4) \text{ for all } x \in X \text{ and for all } y \in A \text{, } x - y \in A \Rightarrow x \in A. \]

Proposition 2.4 (Jun et al., 2005) Let \(X\) be a subtraction algebra and \(x, y \in X\). If \(w \in X\) is an upper bound for \(x\) and \(y\), then the element \(x \lor y = w - ((w - y) - x)\) is the least upper bound for \(x\) and \(y\).

Definition 2.5 (Jun et al., 2005) Let \(Y\) be a non-empty subset of \(X\) then \(Y\) is called a subalgebra of \(X\) if \(x - y \in Y\), whenever \(x, y \in Y\).

Definition 2.6 (Lee and Park, 2007) Let \(f\) be a fuzzy subset of \(X\). Then \(f\) is called a fuzzy subalgebra of \(X\) if it satisfies:

\[ (FS) \ f(x - y) \geq \min\{f(x), f(y)\}, \text{ whenever } x, y \in X. \]

Definition 2.7 (Lee and Park, 2007) A fuzzy subset \(f\) is said to be a fuzzy ideal of \(X\) if it satisfies:

\[ (FI1) \ f(x - y) \geq f(x), \]
\[ (FI2) \text{ if there exists } x \lor y \text{ then } f(x \lor y) \geq \min\{f(x), f(y)\}, \text{ for all } x, y \in X. \]
3. \((\epsilon, \in \vee q_k)\) -intuitionistic fuzzy ideals

In this section we will discuss some properties related to \((\epsilon, \in \vee q_k)\) -intuitionistic fuzzy ideals of subtraction algebras.

**Definition 3.1** An intuitionistic fuzzy set \(A\) in \(X\) is an object of the form \(A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}\), where the function \(\mu_A : X \rightarrow [0,1]\) and \(\gamma_A : X \rightarrow [0,1]\) denote the degree of membership and degree of non-membership of each element \(x \in X\), and \(0 \leq \mu_A(x) + \gamma_A(x) \leq 1\) for all \(x \in X\). For simplicity, we will use the symbol \(A = (\mu_A, \gamma_A)\) for the intuitionistic fuzzy set \(A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}\). We define \(0(x) = 0\) and \(1(x) = 1\) for all \(x \in X\).

**Definition 3.2** Let \(X\) be a subtraction algebra. An intuitionistic fuzzy set \(A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}\), of the form

\[
\chi_{(\alpha, \beta)}^y = \begin{cases} 
(\alpha, \beta) & \text{if } y = x \\
(0, 1) & \text{if } y \neq x
\end{cases}
\]

is said to be an intuitionistic fuzzy point with support \(x\) and value \((\alpha, \beta)\) and is denoted by \(\chi_{(\alpha, \beta)}^y\). A fuzzy point \(\chi_{(\alpha, \beta)}^y\) is said to intuitionistic belongs to (resp., intuitionistic quasi-coincident) with intuitionistic fuzzy set \(A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}\) written \(\chi_{(\alpha, \beta)}^y \in A\) (resp., \(\chi_{(\alpha, \beta)}^y \in A\)) if \(\mu_A(x) \geq \alpha\) and \(\gamma_A(x) \leq \beta\) (resp., \(\mu_A(x) + \alpha > 1\) and \(\gamma_A(x) + \beta < 1\)).

By the symbol \(\chi_{(\alpha, \beta)}^y \in A\) we mean \(\mu_A(x) \geq \alpha + k > 1\) and \(\gamma_A(x) \leq \beta + k < 1\), where \(k \in (0,1)\).

We use the symbol \(\chi_{(\alpha, \beta)}^y \in A\) implies \(\mu_A(x) \geq t\) and \(\gamma_A(x) \leq t\), in the whole paper.

**Definition 3.3** An intuitionistic fuzzy set \(A = (\mu_A, \gamma_A)\) of \(X\) is said to be an \((\epsilon, \in \vee q_k)\) -intuitionistic fuzzy subalgebra of \(X\) if \(\chi_{(\alpha, \beta)}^y \in A\) and \(\gamma_{(\alpha, \beta)}^y \in A\) for all \(x, y \in X, t_1, t_2, t_3, t_4, k \in (0,1)\).

OR

An intuitionistic fuzzy set \(A = (\mu_A, \gamma_A)\) of \(X\) is said to be an \((\epsilon, \in \vee q_k)\) -intuitionistic fuzzy subalgebra of \(X\) if it satisfy the following conditions,

(i) \(\chi_{(\alpha, \beta)}^y \in A\), \(\gamma_{(\alpha, \beta)}^y \in A\) and \(\chi_{(\alpha, \beta)}^z \in A\) for all \(x, y, z \in X, t_1, t_2, t_3, t_4, k \in (0,1)\).

(ii) \[\epsilon \leq \gamma_A(x, y, z) = \frac{\epsilon + \gamma_A(x) \wedge [\eta_A(x)] \wedge \gamma_A(x) \wedge [\gamma_A(x)]}{1 - \epsilon}, \text{ for all } x, y, z \in X, t_1, t_2, t_3, t_4, k \in (0,1)\].

**Example 3.4** Let \(X = \{0, a, b\}\) be a subtraction algebra with the following Cayley table

<table>
<thead>
<tr>
<th>(X)</th>
<th>0</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mu_A(x))</td>
<td>0.5</td>
<td>0.6</td>
<td>0.7</td>
</tr>
<tr>
<td>(\gamma_A(x))</td>
<td>0.1</td>
<td>0.2</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Let us define the intuitionistic fuzzy set \(A = (\mu_A, \gamma_A)\) of \(X\) as

\[
\begin{array}{c|c|c|c|}
\hline
X & 0 & a & b \\
\hline
\mu_A(x) & 0.5 & 0.6 & 0.7 \\
\gamma_A(x) & 0.1 & 0.2 & 0.3 \\
\hline
\end{array}
\]

and

\[
\begin{array}{c|c|c|}
\hline
X & 0 & a & b \\
\hline
\mu_A(x) & 0.5 & 0.6 & 0.7 \\
\gamma_A(x) & 0.1 & 0.2 & 0.3 \\
\hline
\end{array}
\]

then \(A = (\mu_A, \gamma_A)\) is an \((\epsilon, \in \vee q_k)\) -intuitionistic fuzzy subalgebra of \(X\).

**Definition 3.5** An intuitionistic fuzzy set \(A = (\mu_A, \gamma_A)\) of \(X\) is said to be an \((\epsilon, \in \vee q_k)\) -intuitionistic fuzzy ideal of \(X\) if it satisfy the following conditions,

(i) \(\chi_{(\alpha, \beta)}^y \in A\), \(\gamma_{(\alpha, \beta)}^y \in A\) and \(\chi_{(\alpha, \beta)}^z \in A\) for all \(x, y, z \in X, t_1, t_2, t_3, t_4, k \in (0,1)\).

(ii) \(\chi_{(\alpha, \beta)}^y \in A\) and \(\gamma_{(\alpha, \beta)}^y \in A\) for all \(x, y, z \in X, t_1, t_2, t_3, t_4, k \in (0,1)\).

An intuitionistic fuzzy set \(A = (\mu_A, \gamma_A)\) of \(X\) is said to be an \((\epsilon, \in \vee q_k)\) -intuitionistic fuzzy ideal of \(X\) if it satisfy the following conditions,

(i) \(\chi_{(\alpha, \beta)}^y \in A\), \(\gamma_{(\alpha, \beta)}^y \in A\) and \(\chi_{(\alpha, \beta)}^z \in A\) for all \(x, y, z \in X, t_1, t_2, t_3, t_4, k \in (0,1)\).

(ii) \(\chi_{(\alpha, \beta)}^y \in A\) and \(\gamma_{(\alpha, \beta)}^y \in A\) for all \(x, y, z \in X, t_1, t_2, t_3, t_4, k \in (0,1)\).

**Example 3.6** Let \(X = \{0, a, b\}\) be a subtraction algebra with the Cayley table define in Example 3.4, and let us define an intuitionistic fuzzy set \(A = (\mu_A, \gamma_A)\) of \(X\) as

\[
\begin{array}{c|c|c|c|}
\hline
X & 0 & a & b \\
\hline
\mu_A(x) & 0.5 & 0.6 & 0.7 \\
\gamma_A(x) & 0.1 & 0.2 & 0.3 \\
\hline
\end{array}
\]
then $A = (\mu_1, \gamma_1)$ is an $(\varepsilon, \varepsilon \vee q_{0.2})$-intuitionistic fuzzy ideal of $X$.

**Theorem 3.7** An intuitionistic fuzzy set $A = (\mu, \gamma)$ of $X$ is said to be an $(\varepsilon, \varepsilon \vee q_{0.2})$-intuitionistic fuzzy subalgebra of $X$ if and only if $\mu_1(x-y) \geq \min\{\mu_1(x), \mu_1(y), \frac{1-k}{2}\}$ and $\gamma_1(x-y) \leq \max\{\gamma_1(x), \gamma_1(y), \frac{1-k}{2}\}$.

**Proof.** Let $A = (\mu_1, \gamma_1)$ be an $(\varepsilon, \varepsilon \vee q_{0.2})$-intuitionistic fuzzy subalgebra of $X$, and assume on the contrary that there exist some $t \in (0, 1]$ and $r \in [0, 1)$ such that

$$\mu_1(x-y) < t < \min\{\mu_1(x), \mu_1(y), \frac{1-k}{2}\}$$

and

$$\gamma_1(x-y) > r > \max\{\gamma_1(x), \gamma_1(y), \frac{1-k}{2}\}.$$ 

This implies that

$$\mu_1(x) \geq t, \mu_1(y) \geq t, \mu_1(x-y) < t$$

$$\Rightarrow \mu_1(x-y) + t + k \frac{1}{2} \frac{1}{2} + k = 1$$

$$\Rightarrow (x-y) \in q_{0.2}\mu_1,$$

which is a contradiction. Hence

$$\mu_1(x-y) \geq \min\{\mu_1(x), \mu_1(y), \frac{1-k}{2}\}.$$ 

Also from

$$\gamma_1(x-y) > r > \max\{\gamma_1(x), \gamma_1(y), \frac{1-k}{2}\}$$

we get

$$\gamma_1(x) \leq r, \gamma_1(y) \leq r, \gamma_1(x-y) > r$$

$$\Rightarrow \gamma_1(x-y) + r + k \frac{1}{2} \frac{1}{2} + k = 1$$

$$\Rightarrow \frac{r}{x-y} \in q_{0.2}[\gamma_1(x),$$

which is a contradiction. Hence

$$\gamma_1(x-y) \leq \max\{\gamma_1(x), \gamma_1(y), \frac{1-k}{2}\}.$$ 

Conversely, let $\mu_1(x-y) \geq \min\{\mu_1(x), \mu_1(y), \frac{1-k}{2}\}$ and $\gamma_1(x-y) \leq \max\{\gamma_1(x), \gamma_1(y), \frac{1-k}{2}\}$. Let $x_i \in \mu_1$, $y_i \in \mu_1$ for all $x, y \in X$, $t_i, t_2, k \in [0, 1]$ and $\frac{r}{x-y} \in q_{0.2}\gamma_1$ for all $x, y \in X$, $t_i, t_2, k \in (0, 1)$. This implies that

$$\mu_1(x) \geq t_i, \mu_1(y) \geq t_2$$

and

$$\gamma_1(x) \leq t_i, \gamma_1(y) \leq t_2.$$ 

Consider

$$\mu_1(x-y) \geq \min\{\mu_1(x), \mu_1(y), \frac{1-k}{2}\} \geq \min\{t_i, t_2, \frac{1-k}{2}\}.$$ 

If $t_i \lor t_2 \leq \frac{1-k}{2}$, then $\mu_1(x-y) \leq \frac{1-k}{2}$. So $\mu_1(x-y) + (t_i \land t_2) + k \leq \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, which implies that $(x-y) \in q_{0.2}\mu_1$. If $t_i \lor t_2 \leq \frac{1-k}{2}$, then $\mu_1(x-y) \geq t_i \land t_2$. So $(x-y) \in q_{0.2}\mu_1$.

Thus $(x-y) \in q_{0.2}\mu_1$. Also consider

$$\gamma_1(x-y) \leq \max\{\gamma_1(x), \gamma_1(y), \frac{1-k}{2}\} \leq \max\{t_i, t_2, \frac{1-k}{2}\}.$$ 

If $t_i \lor t_2 \leq \frac{1-k}{2}$, then $\gamma_1(x-y) \leq \frac{1-k}{2}$. So $\gamma_1(x-y) + (t_i \land t_2) + k \leq \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, which implies that $\frac{r}{x-y} \in q_{0.2}[\gamma_1(x)$.

If $t_i \lor t_2 > \frac{1-k}{2}$, then $\gamma_1(x-y) \leq t_i \lor t_2$. So $\frac{r}{x-y} \in q_{0.2}[\gamma_1(x$. Hence $A = (\mu_1, \gamma_1)$ is an $(\varepsilon, \varepsilon \vee q_{0.2})$-intuitionistic fuzzy subalgebra of $X$.

**Theorem 3.8** An intuitionistic fuzzy set $A = (\mu_1, \gamma_1)$ of $X$ is said to be an $(\varepsilon, \varepsilon \vee q_{0.2})$-intuitionistic fuzzy ideal of $X$ if and only if

(i) $\mu_1(x-y) \geq \min\{\mu_1(x), \frac{1-k}{2}\}$;

(ii) $\gamma_1(x-y) \leq \max\{\gamma_1(x), \frac{1-k}{2}\}$;

(iii) $\mu_1(x \lor y) \geq \min\{\mu_1(x), \mu_1(y), \frac{1-k}{2}\}$;

(iv) $\gamma_1(x \lor y) \leq \max\{\gamma_1(x), \gamma_1(y), \frac{1-k}{2}\}$.

**Proof.** The proof is similar to the proof of the Theorem 3.7. Proposition Every $(\varepsilon, \varepsilon \vee q_{0.2})$-intuitionistic fuzzy ideal of $X$ is an $(\varepsilon, \varepsilon \vee q_{0.2})$-intuitionistic fuzzy subalgebra of $X$, but converse is not true.

**Example 3.9** Let $X = \{0, a, b\}$ be a subtraction algebra with the Cayley table define in Example 3.4, and let

<table>
<thead>
<tr>
<th>$X$</th>
<th>0</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1(x)$</td>
<td>0.9</td>
<td>0.3</td>
<td>0.6</td>
</tr>
<tr>
<td>$\gamma_1(x)$</td>
<td>0.2</td>
<td>0.6</td>
<td>0.4</td>
</tr>
</tbody>
</table>

then by Theorem 3.7, $A = (\mu_1, \gamma_1)$ is an $(\varepsilon, \varepsilon \vee q_{0.2})$-intuitionistic fuzzy subalgebra of $X$. But by Theorem 3.8, we observe that $A = (\mu_1, \gamma_1)$ is not a $(\varepsilon, \varepsilon \vee q_{0.2})$-intuitionistic fuzzy ideal of $X$. As

$$\mu_1(a \lor b) = \mu_1(0 - ((0 - b) - a)) = \mu_1(0) = 0.9$$

$$\geq \min\{\mu_1(a), \mu_1(b), \frac{1-k}{2}\} = \min\{0.3, 0.6, 0.3\} = 0.3.$$
Definition 3.10 Let \( A = (\mu_A, \gamma_A) \) is an intuitionistic fuzzy set of \( X \). Define the intuitionistic level set as \( A_{(\alpha, \beta)} = \{ x \in X | \mu_A(x) \geq \alpha, \gamma_A(x) \leq \beta \}, \) where \( \alpha \in (0, \frac{1}{2}], \beta \in [\frac{1}{2}, 1) \).

Theorem 3.11 An intuitionistic fuzzy set \( A = (\mu_A, \gamma_A) \) of \( X \) is said to be an \( (\epsilon, \in \vee \gamma) \)-intuitionistic fuzzy subalgebra of \( X \) if and only if \( A_{(\alpha, \beta)} \neq \emptyset \) is a subalgebra of \( X \).

Proof. Let \( A = (\mu_A, \gamma_A) \) be an \( (\epsilon, \in \vee \gamma) \)-intuitionistic fuzzy subalgebra of \( X \). Suppose that \( x, y \in A_{(\alpha, \beta)} \) then \( \mu_A(x) \geq \alpha, \mu_A(y) \geq \alpha \) and \( \gamma_A(x) \leq \beta, \gamma_A(y) \leq \beta \). Since \( A = (\mu_A, \gamma_A) \) is an \( (\epsilon, \in \vee \gamma) \)-intuitionistic fuzzy subalgebra of \( X \), so

\[
\mu_A(x-y) \geq \min\{\mu_A(x), \mu_A(y), 1-k \} \\
\geq \min\{\alpha, \alpha, 1-k \} = \min\{\alpha, 1-k \} = \alpha
\]

and

\[
\gamma_A(x-y) \leq \max\{\gamma_A(x), \gamma_A(y), 1-k \} \\
\leq \max\{\beta, \beta, 1-k \} = \max\{\beta, 1-k \} = \beta.
\]

Thus \( (x-y) \in A_{(\alpha, \beta)} \). Hence \( A_{(\alpha, \beta)} \neq \emptyset \) is a subalgebra of \( X \). Conversely, assume that \( A_{(\alpha, \beta)} \neq \emptyset \) is a subalgebra of \( X \). Assume on contrary that there exist some \( x, y \in X \) such that \( \mu_A(x-y) < \min\{\mu_A(x), \mu_A(y), 1-k \} \) and \( \gamma_A(x-y) > \max\{\gamma_A(x), \gamma_A(y), 1-k \} \). Choose \( \alpha \in (0, \frac{1}{2}], \beta \in [\frac{1}{2}, 1) \) such that \( \mu_A(x-y) < \alpha < \min\{\mu_A(x), \mu_A(y), 1-k \} \) and \( \gamma_A(x-y) > \beta > \max\{\gamma_A(x), \gamma_A(y), 1-k \} \). This implies that \( (x-y) \notin A_{(\alpha, \beta)} \), which is a contradiction to the hypothesis. Hence \( \mu_A(x-y) \geq \min\{\mu_A(x), \mu_A(y), 1-k \} \) and \( \gamma_A(x-y) \leq \max\{\gamma_A(x), \gamma_A(y), 1-k \} \). Thus \( A = (\mu_A, \gamma_A) \) is an \( (\epsilon, \in \vee \gamma) \)-intuitionistic fuzzy subalgebra of \( X \).

Theorem 3.12 An intuitionistic fuzzy set \( A = (\mu_A, \gamma_A) \) of \( X \) is said to be an \( (\epsilon, \in \vee \gamma) \)-intuitionistic fuzzy ideal of \( X \) if and only if \( A_{(\alpha, \beta)} \neq \emptyset \) is an ideal of \( X \).

Proof. The proof is similar to the proof of the Theorem 3.12.

Definition 3.13 Let \( X \) be a subtraction algebra and \( A \subseteq X \), an \( (\epsilon, \in \vee \gamma) \)-intuitionistic characteristic function

\[
\chi_A = \{(x, \mu_{\chi_A}(x), \gamma_{\chi_A}(x)) | x \in S\},
\]

where \( \mu_{\chi_A} \) and \( \gamma_{\chi_A} \) are fuzzy sets respectively, defined as follows:

\[
\mu_{\chi_A} : X \to [0,1] | x \to \mu_{\chi_A}(x) = \begin{cases} \frac{1}{2} & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}
\]

and

\[
\gamma_{\chi_A} : X \to [0,1] | x \to \gamma_{\chi_A}(x) = \begin{cases} 1 & \text{if } x \in A \\ \frac{1}{2} & \text{if } x \notin A \end{cases}
\]

Lemma 3.14 For a non-empty subset \( A \) of a subtraction algebra \( X \), we have

(i) \( A \) is a subalgebra of \( X \) if and only if the characteristic intuitionistic intension set \( \chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle \) of \( A \) in \( X \) is an \( (\epsilon, \in \vee \gamma) \)-intuitionistic fuzzy subalgebra of \( X \). (ii) \( A \) is an ideal of \( X \) if and only if the characteristic intuitionistic intension set \( \chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle \) of \( A \) in \( X \) is an \( (\epsilon, \in \vee \gamma) \)-intuitionistic fuzzy ideal of \( X \).

Proof. The proof is straightforward.

4. \( (\epsilon, \in \vee \gamma) \)-intuitionistic fuzzy soft ideals

Molodtsov defined the notion of a soft set as follows.

Definition 4.1 (Molodtsov, 1999) A pair \( (F, A) \) is called a soft set over \( U \), where \( F \) is a mapping given by \( F : A \to P(U) \). In other words a soft set over \( U \) is a parameterized family of subsets of \( U \).

The class of all intuitionistic fuzzy sets on \( X \) will be denoted by \( IF(X) \).

Definition 4.2 (Maji et al., 2001b) Let \( U \) be an initial universe and \( E \) be the set of parameters. Let \( A \subseteq E \). A pair \( (\tilde{F}, A) \) is called an intuitionistic fuzzy soft set over \( U \), where \( \tilde{F} \) is a mapping given by \( \tilde{F} : A \to IF(U) \).

In general, for every \( \epsilon \in A \), \( \tilde{F}[\epsilon] = \langle \mu_{\tilde{F}[\epsilon]}, \gamma_{\tilde{F}[\epsilon]} \rangle \) is an intuitionistic fuzzy set in \( U \) and it is called intuitionistic fuzzy value set of parameter \( \epsilon \).

Definition 4.3 Let \( U \) be an initial universe and \( E \) be a set of parameters. Suppose that \( A, B \subseteq E \), \( (\tilde{F}, A) \) and \( (\tilde{G}, B) \) are two intuitionistic fuzzy soft sets, we say that \( (\tilde{F}, A) \) is an intuitionistic fuzzy soft subset of \( (\tilde{G}, B) \) if and only if

(1) \( A \subseteq B \),

(2) for all \( \epsilon \in A \), \( \tilde{F}[\epsilon] \) is an intuitionistic fuzzy subset of \( \tilde{G}[\epsilon] \), that is, for all \( x \in U \) and \( \epsilon \in A \), \( \mu_{\tilde{F}[\epsilon]}(x) \leq \mu_{\tilde{G}[\epsilon]}(x) \), and \( \gamma_{\tilde{F}[\epsilon]}(x) \geq \gamma_{\tilde{G}[\epsilon]}(x) \). This relationship is denoted by \( (\tilde{F}, A) \prec (\tilde{G}, B) \).

Definition 4.4 Let \( (\tilde{F}, A) \) and \( (\tilde{G}, B) \) be two intuitionistic fuzzy soft sets over a common universe \( U \). Then
Then the bi-

Let \((\tilde{F}, A)\) and \((\tilde{G}, B)\) be two intuitionistic fuzzy soft sets over a common universe \(U\). Then \((\tilde{F}, A) \vee (\tilde{G}, B)\) is defined by \((\tilde{F}, A) \vee (\tilde{G}, B) = (\tilde{\Theta}, A \times B)\), where \(\tilde{\Theta}[e, x] = \tilde{F}[e] \cup \tilde{G}[e]\) for all \((e, x) \in A \times B\), that is,
\[
\tilde{\Theta}[e, x] = \left\{ \mu_{\tilde{F}[e]}(x) \land \mu_{\tilde{G}[e]}(x), \gamma_{\tilde{F}[e]}(x) \lor \gamma_{\tilde{G}[e]}(x) \right\},
\]
for all \((e, x) \in A \times B, \ x \in U\).

**Definition 4.5** Let \((\tilde{F}, A)\) and \((\tilde{G}, B)\) be two intuitionistic fuzzy soft sets over a common universe \(U\). Then the bi-

Let \((\tilde{F}, A)\) and \((\tilde{G}, B)\) be two intuitionistic fuzzy soft sets over a common universe \(U\). Then the intersection \((\tilde{\Theta}, C)\), where \(C = A \cap B\) and for all \(e \in C\) and \(x \in U\),
\[
\mu_{\tilde{G}[e]}(x) = \begin{cases} 
\mu_{\tilde{F}[e]}(x) & \text{if } e \in A - B \\
\mu_{\tilde{G}[e]}(x) & \text{if } e \in B - A \\
\mu_{\tilde{F}[e]}(x) \land \mu_{\tilde{G}[e]}(x) & \text{if } e \in A \cap B,
\end{cases}
\]
and let \(U\) is the set of cellular brands of mobile companies in the market. Let \(\Theta = \{ \text{attractive, expensive, cheap} \}\) is a parameter space and \(A = \{ \text{attractive, cheap} \}\). Define
\[
\tilde{F} = \{ \text{attractive} \} = \{ (0.0, 0.0, 0.1), (a, 0.8, 0.2), (b, 0.7, 0.3) \},
\]
\[
\tilde{F} = \{ \text{cheap} \} = \{ (0.0, 0.5, 0.12), (a, 0.6, 0.15), (b, 0.7, 0.2) \}.
\]
Let \(k = 0.4\) then \((\tilde{F}, A)\) is an \((e, e \vee q_4)\)-intuitionistic fuzzy soft subalgebra of \(X\).

**Definition 4.12** An intuitionistic fuzzy soft set \((\tilde{F}, A)\) of \(X\) is called an \((e, e \vee q_4)\)-intuitionistic fuzzy soft subalgebra of \(X\), if for all \(e \in A\), \(\tilde{F}[e] = \{ \mu_{\tilde{F}[e]}(x), \gamma_{\tilde{F}[e]}(x) \}\) is an \((e, e \vee q_4)\)-intuitionistic fuzzy soft subalgebra of \(X\), if
\[
(i) \ \mu_{\tilde{F}[e]}(x) = \min\{ \mu_{\tilde{F}[e]}(x), \mu_{\tilde{F}[e]}(y), \frac{1}{2} \},
\]
\[
(ii) \ \gamma_{\tilde{F}[e]}(x) = \max\{ \gamma_{\tilde{F}[e]}(x), \gamma_{\tilde{F}[e]}(y), \frac{1}{2} \},
\]
for all \(x, y \in X\).

**Definition 4.13** An intuitionistic fuzzy soft set \((\tilde{F}, A)\) of \(X\) is called an \((e, e \vee q_4)\)-intuitionistic fuzzy soft ideal of \(X\), if for all \(e \in A\), \(\tilde{F}[e] = \{ \mu_{\tilde{F}[e]}(x), \gamma_{\tilde{F}[e]}(x) \}\) is an \((e, e \vee q_4)\)-intuitionistic fuzzy soft ideal of \(X\), if
\[
(i) \ \mu_{\tilde{F}[e]}(x) = \min\{ \mu_{\tilde{F}[e]}(x), \frac{1}{2} \},
\]
\[
(ii) \ \gamma_{\tilde{F}[e]}(x) = \max\{ \gamma_{\tilde{F}[e]}(x), \frac{1}{2} \},
\]
for all \(x \in X\).
Proposition 4.14 An \((e, e \lor q_k)\)-intuitionistic fuzzy soft ideal of \(X\) is an \((e, e \lor q_k)\)-intuitionistic fuzzy soft subalgebra of \(X\), but converse is not true.

Example 4.15 From Example 4.11, \(\langle \bar{F}, A \rangle\) is not an \((e, e \lor q_k)\)-intuitionistic fuzzy soft ideal of \(X\). As

\[
\mu_{\bar{F}[\text{magic}]}(a \lor b) = \min \{\mu_{\bar{F}[\text{magic}]}(a), \mu_{\bar{F}[\text{magic}]}(b)\},
\]

for all \(a, b \in X\).

Theorem 4.16 Let \(\langle \bar{F}, A \rangle\) and \(\langle \bar{G}, B \rangle\) be two \((e, e \lor q_k)\)-intuitionistic fuzzy soft subalgebras (resp., ideals) of \(X\). Then \(\langle \bar{F}, A \rangle \prec \langle \bar{G}, B \rangle\) is also an \((e, e \lor q_k)\)-intuitionistic fuzzy soft subalgebra (resp., ideal) of \(X\).

Proof. Let \(\langle \bar{F}, A \rangle\) and \(\langle \bar{G}, B \rangle\) be two \((e, e \lor q_k)\)-intuitionistic fuzzy soft ideals of \(X\). We know that \(\langle \bar{F}, A \rangle \prec \langle \bar{G}, B \rangle\) is an \((e, e \lor q_k)\)-intuitionistic fuzzy soft subalgebra of \(X\), where \(\bar{G}[e, e] = \bar{F}[e] \cap \bar{G}[e]\) for all \((e, e) \in A \times B\), that is

\[
\tilde{G}[e, e](x) = \{ \mu_{\bar{F}[e]}(x) \land \mu_{\bar{G}[e]}(x), \gamma_{\bar{F}[e]}(x) \lor \gamma_{\bar{G}[e]}(x) \}
\]

for all \(x \in X\). Let \(x, y \in X\), we have

\[
\mu_{\bar{F}[e]}(x \lor y) = \mu_{\bar{F}[e]}(x) \land \mu_{\bar{G}[e]}(y),
\]

and

\[
\gamma_{\bar{F}[e]}(x \lor y) = \gamma_{\bar{F}[e]}(x) \lor \gamma_{\bar{G}[e]}(y),
\]

Also we have

\[
\begin{align*}
\mu_{\bar{F}[e]}(x) \lor \mu_{\bar{G}[e]}(y) & \geq \min \{\mu_{\bar{F}[e]}(x), \mu_{\bar{G}[e]}(y)\} \land \mu_{\bar{G}[e]}(y), \\
\gamma_{\bar{F}[e]}(x) \lor \gamma_{\bar{G}[e]}(y) & \leq \max \{\gamma_{\bar{F}[e]}(x), \gamma_{\bar{G}[e]}(y)\} \lor \gamma_{\bar{G}[e]}(y),
\end{align*}
\]
Conversely assume that $(\tilde{F}, A)^{(a, b)}$ is a soft subalgebra of $X$. Let there exist some $r \in (0, 1]$ and $s \in (0, 1)$ such that
\[
\mu_{\tilde{F}[x]}(x-y) < r < \min\{\mu_{\tilde{F}[x]}(x), \mu_{\tilde{F}[x]}(y), \frac{1-k}{2}\}
\]
and
\[
\gamma_{\tilde{F}[x]}(x-y) > s > \min\{\gamma_{\tilde{F}[x]}(x), \gamma_{\tilde{F}[x]}(y), \frac{1-k}{2}\}.
\]
This implies that $(x-y) \notin (\tilde{F}, A)^{(a, b)}$, which is contradiction. Thus
\[
\mu_{\tilde{F}[x]}(x-y) \geq \min\{\mu_{\tilde{F}[x]}(x), \mu_{\tilde{F}[x]}(y), \frac{1-k}{2}\}
\]
and
\[
\gamma_{\tilde{F}[x]}(x-y) \leq \max\{\gamma_{\tilde{F}[x]}(x), \gamma_{\tilde{F}[x]}(y), \frac{1-k}{2}\}.
\]
Hence $(\tilde{F}, A)$ is an $(\epsilon, \epsilon \vee q_k)$-intuitionistic fuzzy soft subalgebra of $X$.

**Proposition 4.19** Every $(\epsilon, \epsilon \vee q_k)$-intuitionistic fuzzy soft ideal $(\tilde{F}, A)$ of $X$ satisfies the following,

(i) $\mu_{\tilde{F}[x]}(0) \geq \min\{\mu_{\tilde{F}[x]}(x), \frac{1-k}{2}\}$,

(ii) $\gamma_{\tilde{F}[x]}(0) \leq \max\{\gamma_{\tilde{F}[x]}(x), \frac{1-k}{2}\}$, for all $x \in X$.

**Proof:** By letting $x = y$ in the Definition 4.13, we get the required proof.

**Lemma 4.20** If an $(\epsilon, \epsilon \vee q_k)$-intuitionistic fuzzy soft set $(\tilde{F}, A)$ of $X$ satisfies the followings,

(i) $\mu_{\tilde{F}[x]}(0) \geq \min\{\mu_{\tilde{F}[x]}(x), \frac{1-k}{2}\}$,

(ii) $\gamma_{\tilde{F}[x]}(0) \leq \max\{\gamma_{\tilde{F}[x]}(x), \frac{1-k}{2}\}$,

(iii) $\mu_{\tilde{F}[x]}(x-z) \geq \min\{\mu_{\tilde{F}[x]}((x-y)-z), \mu_{\tilde{F}[x]}(y), \frac{1-k}{2}\}$,

(iv) $\gamma_{\tilde{F}[x]}(x-z) \leq \max\{\gamma_{\tilde{F}[x]}((x-y)-z), \gamma_{\tilde{F}[x]}(y), \frac{1-k}{2}\}$

Then we have $x \leq a \Rightarrow \mu_{\tilde{F}[x]}(x) \geq \min\{\mu_{\tilde{F}[x]}(a), \frac{1-k}{2}\}$ and $\gamma_{\tilde{F}[x]}(x) \leq \max\{\gamma_{\tilde{F}[x]}(a), \frac{1-k}{2}\}$, for all $a, x, y, z \in X$.

**Proof.** Let $a, x \in X$ and $x \leq a$. Consider
\[
\mu_{\tilde{F}[x]}(x) = \mu_{\tilde{F}[x]}(x-0) \text{ by (a2)},
\]
\[
\geq \min\{\mu_{\tilde{F}[x]}((x-a)-0), \mu_{\tilde{F}[x]}(a), \frac{1-k}{2}\} \text{ by (iii)},
\]
\[
= \min\{\mu_{\tilde{F}[x]}(0), \mu_{\tilde{F}[x]}(a), \frac{1-k}{2}\} \text{ by } x \leq a \iff x-a = 0,
\]
\[
= \min\{\mu_{\tilde{F}[x]}(a), \frac{1-k}{2}\} \text{ by (i)}.
\]
Also consider
\[
\gamma_{\tilde{F}[x]}(x) = \gamma_{\tilde{F}[x]}(x-0) \text{ by (a2)},
\]
\[
\leq \max\{\gamma_{\tilde{F}[x]}((x-a)-0), \gamma_{\tilde{F}[x]}(a), \frac{1-k}{2}\} \text{ by (iv)},
\]
\[
= \max\{\gamma_{\tilde{F}[x]}(0), \gamma_{\tilde{F}[x]}(a), \frac{1-k}{2}\} \text{ by } x \leq a \iff x-a = 0,
\]
\[
= \max\{\gamma_{\tilde{F}[x]}(a), \frac{1-k}{2}\} \text{ by (ii)}.
\]
This complete the proof.

**Theorem 4.21** An $(\epsilon, \epsilon \vee q_k)$-intuitionistic fuzzy soft set $(\tilde{F}, A)$ of $X$ satisfies the conditions (i)-(iv) of the Lemma 4.20, if and only if $(\tilde{F}, A)$ is an $(\epsilon, \epsilon \vee q_k)$-intuitionistic fuzzy soft ideal of $X$.

**Proof.** Suppose that $(\tilde{F}, A)$, i.e., $\tilde{F}[x] = \{\mu_{\tilde{F}[x]}(\gamma_{\tilde{F}[x]})\}$ satisfies the conditions (i)-(iv) of the Lemma 4.20. Let $x, y \in X$, then by using (a3) we have $x - y \leq y$. Now by the use of Lemma 4.20,
\[
\mu_{\tilde{F}[x]}(x-y) \geq \min\{\mu_{\tilde{F}[x]}(x), \frac{1-k}{2}\} \text{ and } \gamma_{\tilde{F}[x]}(x-y) \leq \max\{\gamma_{\tilde{F}[x]}(x), \frac{1-k}{2}\} \text{ for all } x, y \in X.
\]
Also, $\mu_{\tilde{F}[x]}(x \vee y) \geq \min\{\mu_{\tilde{F}[x]}(x), \frac{1-k}{2}\}$ and $\gamma_{\tilde{F}[x]}(x \vee y) \leq \max\{\gamma_{\tilde{F}[x]}(x), \frac{1-k}{2}\}$ whenever $x \vee y$ exists in $X$ and by using the Lemma 4.20, we have $\mu_{\tilde{F}[x]}(x \vee y) \geq \min\{\mu_{\tilde{F}[x]}(x), \mu_{\tilde{F}[x]}(y), \frac{1-k}{2}\}$ and $\gamma_{\tilde{F}[x]}(x \vee y) \leq \max\{\gamma_{\tilde{F}[x]}(x), \gamma_{\tilde{F}[x]}(y), \frac{1-k}{2}\}$ for all $x, y \in X$.

Hence $(\tilde{F}, A)$ is an $(\epsilon, \epsilon \vee q_k)$-intuitionistic fuzzy soft ideal of $X$.

Conversely, assume that $(\tilde{F}, A)$ is an $(\epsilon, \epsilon \vee q_k)$-intuitionistic fuzzy soft ideal of $X$. Conditions (i) and (ii) directly follows from the Proposition 4.19, let $x, y, z \in X$. Put $x = y$ and $y = x-z$ in (a3) and (a4). We get from (a3),
\[
(x-z) - y \leq x - z \text{ and from (a4), we have } y-(y-(x-z)) \leq x-z.
\]
Which indicates that an upper bound for $((x-z)-y)$ and $y-(y-(x-z)))$ by using the Proposition 2.4, we have
\[
((x-z)-y) \vee (y-(y-(x-z)))
\]
\[
= x-z.
\]
Thus
\[
\mu_{\tilde{F}[x]}((x-z)-y) = \mu_{\tilde{F}[x]}((x-z)-y) \vee (y-(y-(x-z))))
\]
\[
\geq \min\{\mu_{\tilde{F}[x]}((x-z)-y), \mu_{\tilde{F}[x]}(y-(x-z)), \frac{1-k}{2}\}
\]
\[
= \min\{\mu_{\tilde{F}[x]}((x-z)-y), \mu_{\tilde{F}[x]}(y), \frac{1-k}{2}\} \text{ by (S)} \text{ and (a2) and } y = y.
\]
Which is (iii). And
\[ \gamma_{\tilde{F}_i}(x-z) = \gamma_{\tilde{F}_i}(x-y) \wedge (y-(x-z)) \]
\[ \leq \max \{ \gamma_{\tilde{F}_i}(x-y), \gamma_{\tilde{F}_i}(y-(x-z)), \frac{1-k}{2} \} \]
\[ = \max \{ \gamma_{\tilde{F}_i}(x-y-z), \gamma_{\tilde{F}_i}(y), \frac{1-k}{2} \} \]

Which is (iv). Hence it completes the proof.

**Theorem 4.22** An \((\varepsilon, \in, \nu q^k)\)-intuitionistic fuzzy soft set \((\tilde{F}, A)\) of \(X\) is an \((\varepsilon, \in, \nu q^k)\)-intuitionistic fuzzy soft ideal of \(X\) if and only if it satisfies:

(i) \[ \mu_{\tilde{F}_i}(x-(x-a)-b) \geq \min \{ \mu_{\tilde{F}_i}(a), \mu_{\tilde{F}_i}(b), \frac{1-k}{2} \}, \]

(ii) \[ \gamma_{\tilde{F}_i}(x-(x-a)-b) \leq \max \{ \gamma_{\tilde{F}_i}(a), \gamma_{\tilde{F}_i}(b), \frac{1-k}{2} \} \]

for all \(x, a, b \in X\).

**Proof.** Let \((\varepsilon, \in, \nu q^k)\)-intuitionistic fuzzy set \(A = (\mu_{\tilde{F}_i}, \gamma_{\tilde{F}_i})\) of \(X\) satisfies (i) and (ii). Consider
\[ \mu_{\tilde{F}_i}(x-y) = \mu_{\tilde{F}_i}(x-y-(x-a)-b) \]
\[ \geq \min \{ \mu_{\tilde{F}_i}(x), \mu_{\tilde{F}_i}(y), \frac{1-k}{2} \} \]
\[ = \min \{ \mu_{\tilde{F}_i}(x), \frac{1-k}{2} \} \]

and
\[ \gamma_{\tilde{F}_i}(x-y) = \gamma_{\tilde{F}_i}(x-y-(x-a)-b) \]
\[ \leq \max \{ \gamma_{\tilde{F}_i}(x), \gamma_{\tilde{F}_i}(y), \frac{1-k}{2} \} \]
\[ = \max \{ \gamma_{\tilde{F}_i}(x), \frac{1-k}{2} \} \]

Also suppose that \(x \vee y\) exists in \(X\) and by using the Proposition 2.4, we have \(x \vee y = w-(w-x)\). Now using the given conditions
\[ \mu_{\tilde{F}_i}(x \vee y) = \mu_{\tilde{F}_i}(w-(w-x)-y) \]
\[ \geq \min \{ \mu_{\tilde{F}_i}(w), \mu_{\tilde{F}_i}(y), \frac{1-k}{2} \} \] by (i),

and
\[ \gamma_{\tilde{F}_i}(x \vee y) = \gamma_{\tilde{F}_i}(w-(w-x)-y) \]
\[ \leq \max \{ \gamma_{\tilde{F}_i}(w), \gamma_{\tilde{F}_i}(y), \frac{1-k}{2} \} \] by (ii).

Hence \(A = (\mu_{\tilde{F}_i}, \gamma_{\tilde{F}_i})\) is an \((\varepsilon, \in, \nu q^k)\)-intuitionistic fuzzy soft ideal of \(X\).

Conversely, let \(A = (\mu_{\tilde{F}_i}, \gamma_{\tilde{F}_i})\) be an \((\varepsilon, \in, \nu q^k)\)-intuitionistic fuzzy soft ideal of \(X\). By Theorem 4.21,
\[ \mu_{\tilde{F}_i}(x-z) \geq \min \{ \mu_{\tilde{F}_i}(x-y-z), \mu_{\tilde{F}_i}(y), \frac{1-k}{2} \} \]

and
\[ \gamma_{\tilde{F}_i}(x-z) \leq \max \{ \gamma_{\tilde{F}_i}(x-y-z), \gamma_{\tilde{F}_i}(y), \frac{1-k}{2} \} \].

Let \(z = (x-a)-b\) and \(y = b\), then
\[ \mu_{\tilde{F}_i}(x-(x-a)-b) \geq \min \{ \mu_{\tilde{F}_i}(a), \mu_{\tilde{F}_i}(b), \frac{1-k}{2} \} \]
\[ = \min \{ \mu_{\tilde{F}_i}(a), \mu_{\tilde{F}_i}(b), \frac{1-k}{2} \} \] by (S) and (a2) and \(x = y\). Hence \(A = (\mu_{\tilde{F}_i}, \gamma_{\tilde{F}_i})\) satisfy condition (i) and (ii).

**Lemma 4.23** An intuitionistic fuzzy set \(A = (\mu_{\tilde{F}_i}, \gamma_{\tilde{F}_i})\) of \(X\) is an \((\varepsilon, \in, \nu q^k)\)-intuitionistic fuzzy soft ideal of \(X\) if and only if fuzzy sets \(\mu_{\tilde{F}_i}\) and \(\gamma_{\tilde{F}_i}\) are \((\varepsilon, \in, \nu q^k)\)-fuzzy soft ideals of \(X\).

**Proof.** The proof is straightforward.

**Theorem 4.24** An intuitionistic fuzzy set \(A = (\mu_{\tilde{F}_i}, \gamma_{\tilde{F}_i})\) of \(X\) is an \((\varepsilon, \in, \nu q^k)\)-intuitionistic fuzzy soft ideal of \(X\) if \(\triangleright A = (\mu_{\tilde{F}_i}, \mu_{\tilde{F}_i})\) and only if \(\triangleleft A = (\gamma_{\tilde{F}_i}, \gamma_{\tilde{F}_i})\) are \((\varepsilon, \in, \nu q^k)\)-intuitionistic fuzzy soft ideal of \(X\).

**Proof.** The proof is straightforward.

**Theorem 4.25** If \(A = (\mu_{\tilde{F}_i}, \gamma_{\tilde{F}_i})\) is an \((\varepsilon, \in, \nu q^k)\)-intuitionistic fuzzy soft ideal of \(X\) then
\[ X_{\mu_{\tilde{F}_i}} = \{ x \in X : \mu_{\tilde{F}_i}(x) = \min \{ \mu_{\tilde{F}_i}(x), \frac{1-k}{2} \} = \mu_{\tilde{F}_i}(0) \} \]
and
\[ X_{\gamma_{\tilde{F}_i}} = \{ x \in X : \gamma_{\tilde{F}_i}(x) = \max \{ \gamma_{\tilde{F}_i}(x), \frac{1-k}{2} \} = \gamma_{\tilde{F}_i}(0) \} \]
are soft ideals of \(X\).

**Proof.** Let \(a \in X_{\mu_{\tilde{F}_i}}\) and \(x \in X\). Then by definition \(\min \{ \mu_{\tilde{F}_i}(x), \frac{1-k}{2} \} = \mu_{\tilde{F}_i}(0)\). Since \(A = (\mu_{\tilde{F}_i}, \gamma_{\tilde{F}_i})\) is an \((\varepsilon, \in, \nu q^k)\)-intuitionistic fuzzy soft ideal of \(X\) so we have
(i) \[ \mu_{\tilde{F}_i}(a-x) \geq \min \{ \mu_{\tilde{F}_i}(a), \frac{1-k}{2} \} = \mu_{\tilde{F}_i}(0) \]
which implies that \((a-x) \in X_{\mu_{\tilde{F}_i}}\).

(ii) \[ \gamma_{\tilde{F}_i}(a-x) \leq \max \{ \gamma_{\tilde{F}_i}(a), \frac{1-k}{2} \} = \gamma_{\tilde{F}_i}(0) \]
which implies that \((a-x) \in X_{\gamma_{\tilde{F}_i}}\).

Let \(x \vee y\) exists in \(X\), we get
we have
\[ G^{-}\text{and}\ G^{+}\text{-intuitionistic fuzzy soft subalgebras of } X. \]

**Theorem 4.26** Let \( \tilde{F}, A \) and \( \tilde{G}, B \) be two \( (\varepsilon, \varepsilon \lor q_k) \)-intuitionistic fuzzy soft subalgebras (resp., ideals) of \( X \). Then so is \( \tilde{F}, A \cap \tilde{G}, B \).

**Proof.** Let \( \tilde{F}, A \) and \( \tilde{G}, B \) be two \( (\varepsilon, \varepsilon \lor q_k) \)-intuitionistic fuzzy soft subalgebras of \( X \). We know that \( \tilde{F}, A \cap \tilde{G}, B = (\tilde{G}, C) \), where \( C = A \cup B \). Now for any \( \varepsilon \in C \) and \( x, y \in X \), we consider the following cases

- **Case 1:** For any \( \varepsilon \in A - B \), we have
  \[
  \mu_{\tilde{F}, \varepsilon}(x \lor y) = \mu_{\tilde{F}, \varepsilon}(x \lor y) \geq \min\{\mu_{\tilde{F}, \varepsilon}(x), \mu_{\tilde{F}, \varepsilon}(y), 1 - \frac{k}{2}\} = \min\{\mu_{\tilde{F}, \varepsilon}(x), \mu_{\tilde{F}, \varepsilon}(y), 1 - \frac{k}{2}\},
  \]
  and
  \[
  \gamma_{\tilde{F}, \varepsilon}(x \lor y) = \gamma_{\tilde{F}, \varepsilon}(x \lor y) \leq \max\{\gamma_{\tilde{F}, \varepsilon}(x), \gamma_{\tilde{F}, \varepsilon}(y), 1 - \frac{k}{2}\} = \max\{\gamma_{\tilde{F}, \varepsilon}(x), \gamma_{\tilde{F}, \varepsilon}(y), 1 - \frac{k}{2}\}.
  \]

- **Case 2:** For any \( \varepsilon \in B - A \), we have
  \[
  \mu_{\tilde{F}, \varepsilon}(x \lor y) = \mu_{\tilde{F}, \varepsilon}(x \lor y) \geq \min\{\mu_{\tilde{F}, \varepsilon}(x), \mu_{\tilde{F}, \varepsilon}(y), 1 - \frac{k}{2}\} = \min\{\mu_{\tilde{F}, \varepsilon}(x), \mu_{\tilde{F}, \varepsilon}(y), 1 - \frac{k}{2}\},
  \]
  and
  \[
  \gamma_{\tilde{F}, \varepsilon}(x \lor y) = \gamma_{\tilde{F}, \varepsilon}(x \lor y) \leq \max\{\gamma_{\tilde{F}, \varepsilon}(x), \gamma_{\tilde{F}, \varepsilon}(y), 1 - \frac{k}{2}\} = \max\{\gamma_{\tilde{F}, \varepsilon}(x), \gamma_{\tilde{F}, \varepsilon}(y), 1 - \frac{k}{2}\}.
  \]

- **Case 3:** For any \( \varepsilon \in A \cap B \), we have \( \mu_{\tilde{F}, \varepsilon} \cap \mu_{\tilde{G}, \varepsilon} \) and \( \gamma_{\tilde{F}, \varepsilon} \cup \gamma_{\tilde{G}, \varepsilon} \). Analogously to the proof of Theorem 4.16. Hence \( \tilde{F}, A \cap \tilde{G}, B \) is an \( (\varepsilon, \varepsilon \lor q_k) \)-intuitionistic fuzzy soft subalgebra of \( X \).

**Theorem 4.27** Let \( \tilde{F}, A \) and \( \tilde{G}, B \) be two \( (\varepsilon, \varepsilon \lor q_k) \)-intuitionistic fuzzy soft subalgebras (resp., ideals) of \( X \). Then so is \( \tilde{F}, A \cup \tilde{G}, B \).

**Proof.** The proof is similar to the proof of the Theorem 4.26.

**Acknowledgements**

We highly appreciate the detailed valuable comments of the referees which greatly improve the quality of this paper.

**References**


