On fuzzy $b$-locally open sets in bitopological spaces

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Abstract

In this article we introduce the notion of fuzzy $b$-locally open ($bLO$) sets, fuzzy $bLO^*$ sets, fuzzy $bLO^{**}$ sets in fuzzy bitopological spaces and obtain several characterizations and some properties of these sets. Also we introduce the notion of fuzzy $b$-locally continuous functions on bitopological spaces.

Keywords: fuzzy bitopological spaces, fuzzy $bLO$ sets, fuzzy $bLO^*$ sets, fuzzy $bLO^{**}$ sets.

1. Introduction and Preliminaries

The notion of fuzzy sets was introduced by L.A. Zadeh in 1965, and thereafter the paper of Chang (1968) paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. The notion of fuzzyness has been applied for studying different aspects of mathematics by Tripathy and Baruah (2010); Tripathy and Borgohain (2011); Tripathy et al. (2013); Tripathy and Ray (2012); Tripathy and Sarma (2012a) and many workers on sequence spaces in recent years. The notion of bitopological spaces has been investigated from different aspects by Tripathy and Acharjee (2014); Tripathy and Debnath (2013); Tripathy and Sarma (2011; 2012; 2013; 2014) and others. Kandil (1989) introduced the concept of fuzzy bitopological spaces. Later on several authors were attracted by the notion of fuzzy bitopological spaces. The notion of $b$-locally open sets in bitopological spaces was introduced by Tripathy and Sarma (2011). In this paper we introduce the concept of $b$-locally open sets in fuzzy bitopological spaces.

Let $(X, \tau)$ be a topological space. Then

**Definition 1.1 [Andrijevic (1996)].** Let $A \subseteq X$, then $A$ is said to be $b$-open if $A \subseteq cl(intA) \cap int(clA)$, where $cl(A)$ and $int(A)$ denote the closure and interior of the set $A$.

**Definition 1.2 [Kuratowski and Sierpinski (1921)].** Let $A \subseteq X$, then $A$ is said to be locally closed if $A=G \cap F$, where $G$ is an open set in $X$ and $F$ is closed in $X$.

**Definition 1.3 [Nasef (2001)].** Let $A \subseteq X$, then $A$ is said to be $b$-locally closed if $A=G \cap F$, where $G$ is $b$-open set in $X$ and $F$ is $b$-closed in $X$.

**Definition 1.4 [Tripathy and Sarma (2011)].** A subset $A$ of a bitopological space $(X, \tau, \tau)$ is called $(\tau, \tau)$-locally open (in short $(\tau, \tau)$-LO) if $A=G \cup F$, where $G$ is $\tau$-closed and $F$ is $\tau$-open in $(X, \tau, \tau)$.

**Definition 1.5 [Tripathy and Sarma (2011)].** A subset $A$ of a space $(X, \tau, \tau)$ is called $(\tau, \tau)$-locally open( in short $(\tau, \tau)$-bLO) if $A=G \cup F$, where $G$ is $\tau$-$b$-closed and $F$ is $b$-open in $(X, \tau, \tau)$.

**Definition 1.6 [Tripathy and Sarma (2011)].** A subset $A$ of a space $(X, \tau, \tau)$ is called $(\tau, \tau)$-$bLO^{**}$ if there exists a $\tau$-$b$-closed set $G$ and a $\tau$-$b$-open set $F$ of
Thus we have, \( a = \beta \vee \gamma \) where \( \beta \) is \( \tau \)-fuzzy \( b \)-closed and 
\( \gamma \) is \( \tau \)-fuzzy open.

Hence \( \alpha \in (\tau, \tau)-FbLO\). 

(b) Since \( \alpha \in (\tau, \tau)-FLO(X) \), then there exists a \( \tau \)-fuzzy closed set \( \beta \) and a \( \tau \)-fuzzy open set \( \gamma \) such that \( \alpha = \beta \vee \gamma \).

Since \( \gamma \) is \( \tau \)-fuzzy open, we have \( \gamma \subseteq \operatorname{cl}(\gamma) \) and
\( \gamma \subseteq \operatorname{int}(\gamma) \). 

Hence \( \gamma \subseteq \operatorname{cl}(\gamma) \) \( \cap \) \( \operatorname{int}(\gamma) \). 

Thus \( \gamma \) is \( \tau \)-fuzzy \( b \)-open.

Remark 2.6

The converse is not necessarily true. It is clear from the following example:

Example 2.7

Let \( X = \{a, b, c\} \) and consider the fuzzy sets on \( X \) are
\( a = \{0.9, 0.8, 0.7\} \), \( b = \{0.3, 0.5, 0.2\} \), \( c = \{0.1, 0.2, 0.3\} \), and \( a = \{0.1, 0.2, 0.3\} \). 

Let \( \tau = \{0, 1\, \alpha, \alpha, \alpha \vee \gamma, \alpha, \alpha \vee \gamma\} \) and \( \tau = \{0, 1, 1, 1, \alpha\} \) be two fuzzy topologies on \( X \).

Then for \( \tau \), \( \operatorname{cl}(a) = \alpha' \), \( \operatorname{int}(\operatorname{cl}(a)) = a' \), \( \operatorname{int}(\operatorname{a}) = a' \), \( \operatorname{int}(\operatorname{a}) = a' \).

Thus \( \alpha \) is \( \tau \)-fuzzy closed set in \( (X, \tau, \tau) \).

Next let \( \alpha = \{0, 1, 1, 1, \alpha\} \) and \( \alpha = \{0, 1, 1, 1, \alpha\} \). 

Thus \( \alpha \) is \( \tau \)-fuzzy \( b \)-open in \( (X, \tau, \tau) \).

Next let \( \alpha = \{0, 1, 1, 1, \alpha\} \) and \( \alpha = \{0, 1, 1, 1, \alpha\} \). 

Thus \( \alpha \) is \( \tau \)-fuzzy open set in \( (X, \tau, \tau) \).

Theorem 2.8

Let \( \alpha \) be a fuzzy subset of a fuzzy bitopological space \( (X, \tau, \tau) \). If \( \alpha \in (\tau, \tau)-FbLO(X) \), then \( \alpha \in (\tau, \tau)-FbLO(X) \).

Proof.

Let \( \alpha \in (\tau, \tau)-FbLO(X) \), then there exists a \( \tau \)-fuzzy \( b \)-closed set \( \beta \) and a \( \tau \)-fuzzy open set \( \gamma \) such that \( \alpha = \beta \vee \gamma \).

Since \( \gamma \) is \( \tau \)-fuzzy open, we have \( \gamma \subseteq \operatorname{cl}(\gamma) \) and \( \gamma \subseteq \operatorname{int}(\gamma) \).

Thus \( \gamma \subseteq \operatorname{cl}(\gamma) \) \( \cap \) \( \operatorname{int}(\gamma) \). 

Hence \( \gamma \subseteq \operatorname{cl}(\gamma) \) \( \cap \) \( \operatorname{int}(\gamma) \). 

Thus \( \gamma \) is \( \tau \)-fuzzy \( b \)-open set in \( (X, \tau, \tau) \).

Remark 2.9

The converse of the above theorem is not always true. It follows from the following example:
Example 2.10
Let \( X = \{a, b, c\} \) and consider the fuzzy sets on \( X \),
\[ 
\alpha = \{a_0, b_0, c_0\}, \quad \alpha = \{a_0, b_0, c_0\}, \quad \alpha = \{a_0, b_0, c_0\}, \quad \beta = \{a_0, b_0, c_0\}.
\]
Let \( \tau = \tau = \{0, 1\} \), \( \alpha, \beta, \alpha \lor \alpha, \alpha \land \alpha \) be fuzzy topologies on \( X \).

Then \( \alpha \) is a \( \tau \)-fuzzy \( b \)-closed set and \( \beta \) is a \( \tau \)-fuzzy \( b \)-open set. Thus \( \lambda = \alpha \lor \beta \lor (\tau, \tau) \)-FbLO\((X)\) but \( \lambda \not\in (\tau, \tau) \)-FbLO\(*\)(X).

**Proof:**
Can be established following standard techniques.

**Theorem 2.11**
Let \( \alpha \) be a fuzzy subset of a fuzzy bitopological space \( (X, \tau, \tau) \). If \( \alpha \in (\tau, \tau) \)-FbLO\((X)\) then \( \alpha \in (\tau, \tau) \)-FbLO\(*\)(X).

**Proof:**
Can be established following standard techniques.

**Remark 2.12**
The converse of the above theorem is not always true. It follows from the following example:

**Example 2.13**
Let \( X = \{a, b, c\} \) and consider the fuzzy sets on \( X \) are
\[ 
\alpha = \{a_0, b_0, c_0\}, \quad \alpha = \{a_0, b_0, c_0\}, \quad \alpha = \{a_0, b_0, c_0\}, \quad \beta = \{a_0, b_0, c_0\}.
\]

Let \( \tau = \tau = \{0, 1\} \), \( \alpha, \beta, \alpha \lor \alpha, \alpha \land \alpha \) be a fuzzy topologies on \( X \).

Then \( \alpha \) is a \( \tau \)-fuzzy \( b \)-closed set and \( \beta \) is a \( \tau \)-fuzzy \( b \)-open set. Thus \( \lambda = \alpha \lor \beta \lor (\tau, \tau) \)-FbLO\((X)\) but \( \lambda \not\in (\tau, \tau) \)-FbLO\(*\)(X) because \( \alpha \) is not \( \tau \)-fuzzy closed in \( X \).

**Theorem 2.14**
Let \( \alpha \) and \( \beta \) be any two fuzzy subsets of a fuzzy bitopological space \( (X, \tau, \tau) \). If \( \alpha \in (\tau, \tau) \)-FbLO\((X)\) and \( \beta \) is \( \tau \)-fuzzy \( b \)-closed and \( \tau \)-fuzzy \( b \)-open, then \( \alpha \lor \beta \in (\tau, \tau) \)-FbLO\((X)\).

**Proof:**
Since \( \alpha \in (\tau, \tau) \)-FbLO\((X)\), then there exists a \( \tau \)-fuzzy \( b \)-closed set \( \alpha \) and a \( \tau \)-fuzzy \( b \)-open set \( \alpha \) such that \( \alpha = \alpha \lor \alpha \).

We have, \( \alpha \lor \beta = (\alpha \lor \alpha) \lor \beta = (\alpha \lor \beta) \lor (\alpha \lor \beta) \). Since \( \beta \) is \( \tau \)-fuzzy \( b \)-closed and \( \tau \)-fuzzy \( b \)-open, then \( \alpha \lor \beta \) is \( \tau \)-fuzzy \( b \)-closed.

Thus there exists a \( \tau \)-fuzzy \( b \)-closed set \( \alpha \lor \beta \) and \( \tau \)-fuzzy \( b \)-open set \( \alpha \lor \beta \) such that \( \alpha \lor \beta = (\alpha \lor \beta) \lor (\alpha \lor \beta) \). Hence \( \alpha \lor \beta \in (\tau, \tau) \)-FbLO\((X)\).

**Theorem 2.15**
Let \( \alpha \in (\tau, \tau) \)-FbLO\((X)\) and \( \beta \) be a \( \tau \)-fuzzy closed and \( \tau \)-fuzzy open subsets of \( (X, \tau, \tau) \), then \( \alpha \lor \beta \in (\tau, \tau) \)-FbLO\(*\)(X).

**Proof:**
Can be established following standard techniques.

**Theorem 2.16**
Let \( \alpha \in (\tau, \tau) \)-FbLO\(*\)(X) and \( \beta \) be a \( \tau \)-fuzzy closed and \( \tau \)-fuzzy open subsets of \( (X, \tau, \tau) \), then \( \alpha \lor \beta \in (\tau, \tau) \)-FbLO\(*\)(X).

**Proof:**
Can be established following standard techniques.
Theorem 2.21
If \( \alpha, \beta \in (\tau_1, \tau_2)\)-FbLO \((X)\) then \( \alpha \vee \beta \in (\tau_1, \tau_2)\)-FbLO \((X)\).

Proof:
Let \( \alpha, \beta \in (\tau_1, \tau_2)\)-FbLO \((X)\). Then there exist \( \tau_i\)-fuzzy \( b\)-closed sets \( \alpha_i, \beta_i \) and \( \tau_j\)-fuzzy \( b\)-open sets \( \alpha_j, \beta_j \) such that \( \alpha = \alpha_1 \vee \alpha_2 \) and \( \beta = \beta_1 \vee \beta_2 \).

We have \( \alpha \vee \beta = (\alpha \vee \alpha_1) \vee (\beta \vee \beta_1) = (\alpha_1 \vee \beta_1) \vee (\alpha_2 \vee \beta_2) \),
where \( (\alpha \vee \alpha_1) \) is \( \tau_i\)-fuzzy \( b\)-closed set and \( (\alpha \vee \beta_1) \) is \( \tau_i\)-fuzzy \( b\)-open set.

Hence \( \alpha \vee \beta \in (\tau_1, \tau_2)\)-FbLO \((X)\).

Theorem 2.22
If \( \alpha, \beta \in (\tau_1, \tau_2)\)-FbLO \(^*\) \((X)\) then \( \alpha \vee \beta \in (\tau_1, \tau_2)\)-FbLO \(^*\) \((X)\).

Proof:
Can be established following standard techniques.

Theorem 2.23
If \( \alpha, \beta \in (\tau_1, \tau_2)\)-FbLO \(^{**}\) \((X)\) then \( \alpha \vee \beta \in (\tau_1, \tau_2)\)-FbLO \(^{**}\) \((X)\).

Proof:
Can be established following standard techniques.

Definition 2.24
Let \((X, \tau_1, \tau_2)\) and \((Y, \rho_1, \rho_2)\) be two bitopological spaces and \( f : X \rightarrow Y \) be a mapping. Then \( f \) is said to be fuzzy locally continuous if the inverse image of each FLO-set of \( Y \) is FLO in \( X \).

Definition 2.25
Let \((X, \tau_1, \tau_2)\) and \((Y, \rho_1, \rho_2)\) be two bitopological spaces and \( f : X \rightarrow Y \) be a mapping. Then \( f \) is said to be fuzzy \( b\)-locally continuous if the inverse image of each FbLO-set of \( Y \) is FbLO in \( X \).

From the definition it is obvious that every fuzzy locally continuous function is fuzzy \( b\)-locally continuous but the converse may not be true (by Theorem 2.5 and Theorem 2.7).

References
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