The $\chi^2$ sequence space over $p$-metric spaces defined by Musielak modulus

Nagarajan Subramanian*, Chinappan Priya 2, and Nallamuthu Saivaraju2

1 Department of Mathematics, Shanmugha Arts, Science, Technology and Research Academy University, Thanjavur-613 401, India.
2 Department of Mathematics, Shri Angalamman College of Engineering and Technology, Trichirappalli-621 105, India

Received: 14 May 2014; Accepted: 1 August 2014

Abstract

In this paper, we introduce bonacci numbers of $\chi^2(\mathcal{F})$ sequence space over $p$–metric spaces defined by Musielak function and examine some topological properties of the resulting these spaces.

Keywords: analytic sequence, double sequences, $\chi^2$ space, fibonacci number, Musielak - modulus function, $p$–metric space

1. Introduction

Throughout $w$, $\chi$ and $\Lambda$ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write $w^2$ for the set of all complex sequences $(x_{mn})$, where $m, n \in \mathbb{N}$, the set of positive integers. Then, $w^2$ is a linear space under the coordinate-wise addition and scalar multiplication.


We procure the following sets of double sequences:

$$\mathcal{M}(t) := \left\{(x_{mn}) \in w^2 : \sup_{m,n} |x_{mn}|^t < \infty \right\},$$

$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : p – \lim_{m,n \to \infty} |x_{mn}|^t = 1 \text{ for some } t \in \mathbb{C} \right\},$$

$$\mathcal{C}_{bp}(t) := \left\{(x_{mn}) \in w^2 : p – \lim_{m,n \to \infty} |x_{mn}|^t = 1 \right\},$$

$$\mathcal{L}_u(t) := \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^t < \infty \right\},$$

$$\mathcal{C}_{bp}(t) = \bigcap \mathcal{M}(t) \text{ and } \mathcal{C}_{bp}(t) = \bigcap \mathcal{M}(t) \text{ for all } m, n \in \mathbb{N};$$

where $t = (t_{mn})$ is the sequence of strictly positive reals $t_{mn}$ for all $m, n \in \mathbb{N}$ and $p – \lim_{m,n \to \infty}$ denotes the limit in the Pringsheim’s sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}(t), \mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{bp}(t)$ reduce to the sets $\mathcal{M}, \mathcal{C}_p, \mathcal{C}_{bp}$ and $\mathcal{C}_{bp}$ respectively. Now, we may summarize the knowledge given in some document’s related to the double sequence spaces. Gokhan et al. (2005) have proved that $\mathcal{M}(t)$ and $\mathcal{C}_p(t)$, $\mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and obtained the $\alpha$, $\beta$, $\gamma$- duals of the spaces $\mathcal{M}(t)$ and $\mathcal{C}_{bp}(t)$ : Quite recently, in her PhD thesis, Zeltser (2001) has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences.

Mursaleen et al. (2003, 2004, 2013, 2014) and Tripathy et al. (2003, 2004, 2006, 2007, 2008, 2009, 2010, 2011 and 2013) have independently introduced the statistical convergence and Cauchy for double sequences and established the relation between statistical convergent and strongly Cesaro summable double sequences. Alay and Bas91sar (2005) have defined the spaces $\mathcal{B}S$, $\mathcal{B}S(t)$, $\mathcal{C}S_p$, $\mathcal{C}S_{bp}$, $\mathcal{C}S_\gamma$ and $\mathcal{B}V$ of

* Corresponding author.

Email address: nsmaths@yahoo.com
The vector space of all double analytic\( \sum_{m,n} a_{mn} x_{mn} \) was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For\( a, b, \psi, \Delta \geq 0 \) and \( 0 < p < 1 \); we have
\[
(a + b)^p \leq a^p + b^p. \tag{1.1}
\]

The double series\( \sum_{m,n} x_{mn} \) is called convergent if and only if the double sequence \((x_{mn})\) is convergent, where \( x_{mn} = \sum_{i,j=m}^{\infty} x_{ij} (m,n \in \mathbb{N}) \).

A sequence \( x = (x_{mn}) \) is said to be double analytic if \( \sup_{m,n} |x_{mn}|^{1/m+n} < \infty \). The vector space of all double analytic sequences will be denoted by \( \Lambda^2 \). A sequence \( x = (x_{mn}) \) is called double gai sequence if \( (m+n)!|x_{mn}|^{1/m+n} \to 0 \) as \( m,n \to \infty \). The double gai sequences will be denoted by \( \chi^2 \). Let \( \phi = \{ \text{all finite sequences} \} \).

Consider a double sequence \( x = (x_{ij}) \). The \((m,n)\)th section \( x^{(m,n)} \) of the sequence is defined by \( x^{(m,n)} = \sum_{i,j=m}^{\infty} x_{ij} \mathcal{J}_{ij} \) for all \( m,n \in \mathbb{N} \); where \( \mathcal{J}_{ij} \) denotes the double sequence whose only non zero term is a \( \frac{1}{(i+j)!} \) in the \((i,j)\)th place for each \( i, j \in \mathbb{N} \).

An FK-space (or a metric space) \( X \) is said to have AK property if \( (\mathcal{J}_{mn}) \) is a Schauder basis for \( X \). Or equivalently \( x^{(m,n)} \to x \).

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings \( x = (x_{ij}) \to (x_{mn}) (m,n \in \mathbb{N}) \) are also continuous.

Let \( M \) and \( \Phi \) be mutually complementary modulus functions. Then, we have
\[(i) \quad \forall u, y \geq 0, \quad uv \leq M(u) + \Phi(v), \quad \text{(Young's inequality)} \]
\[\text{[See Kamthan et al. (1981)]} \tag{1.2}\]
\[(ii) \quad \forall u \geq 0, \quad u\eta(u) = M(u) + \Phi(\eta(u)). \tag{1.3}\]
\[(iii) \quad \forall u \geq 0, \quad 0 < \lambda < 1, \quad M(\lambda u) \leq \lambda M(u). \tag{1.4}\]

Lindenstrauss and Tzafriri (1971) used the idea of Orlicz function to construct Orlicz sequence space
\[
\ell_M = \left\{ x \in w: \sum_{l=1}^{\infty} M\left(\frac{k_l}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},
\]
where the space \( \ell_M \) with the norm
\[
\|x\| = \inf \left\{ \rho > 0: \sum_{l=1}^{\infty} M\left(\frac{k_l}{\rho}\right) \leq 1 \right\},
\]
becomes a Banach space which is called an Orlicz sequence space. For \( M(t) = t^p (1 \leq p < \infty) \), the spaces \( \ell_M \) coincide with the classical sequence space \( \ell_p \).

A sequence \( f = (f_{mn}) \) of modulus function is called a Musielak-modulus function. A sequence \( g = (g_{mn}) \) defined by
\[
g_{mn}(v) = \sup \{|v| u - f_{mn}(u): u \geq 0\}, \quad m, n = 1, 2, ...
\]
is called the complementary function of a Musielak-modulus function \( f \).

For a given Musielak modulus function \( f \), the Musielak-modulus sequence space \( t_f \) is defined by
\[
t_f = \left\{ x \in w: I_f\left(\left|\frac{x_{mn}}{n}\right|^{1/n} \right) \to 0 \text{ as } m, n \to \infty, \right\}
\]
where \( t_f \) is a convex modular defined by
\[
I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}\left(|x_{mn}|^{1/m+n}\right), \quad x = (x_{mn}) \in t_f.
\]

We consider \( t_f \) equipped with the Luxemburg metric
\[
d(x, y) = \sup_{mn} \left\{ \inf \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}\left(|x_{mn}|^{1/m+n}\right) \right) \leq 1 \right\}.
\]

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [16] as follows
2. Definition and Preliminaries

Let $n \in \mathbb{N}$ and $X$ be a real vector space of dimension $w$, where $n \leq w$. A real valued function $d(x_1, \ldots, x_n)$ is a double sequence which is defined as:

$$Z(\Delta) = \{ x = (x_{m,n}) \in w : (\Delta x_{m,n}) \in Z \},$$

for $Z = c, c_0$ and $\ell_p$, where $\Delta x_{m,n} = x_{m,n} - x_{m-1,n}$ for all $k \in \mathbb{N}$

Here $c, c_0$ and $\ell_p$ denote the classes of convergent, null and bounded scalar valued single sequences respectively. The difference sequence space $bv_p$ of the classical space $\ell_p$ is introduced and studied in the case $1 \leq p \leq \infty$ by Basar and Altay and in the case $0 < p < 1$ by Altay et al. (2005). The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and $bv_p$ are Banach spaces normed by

$$\|x\| = \|x_1\| + \sup_{x_{m,n}} |x_{m,n}| + \|x_{m,n}| \| \leq \left( \sum_{m=1}^{\infty} \left| \sum_{n=1}^{\infty} x_{m,n} \right|^p \right)^{1/p} (1 \leq p < \infty).$$

Later on the notion was further investigated by others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{ x = (x_{m,n}) \in w^2 : (\Delta x_{m,n}) \in Z \},$$

where $Z = \Delta^i, \Delta^j$ and $\Delta^i x_{m,n} = (x_{m,n} - x_{m-1,n}) - (x_{m-1,n} - x_{m-1,n+1}) = x_{m,n} - x_{m-1,n+1} + x_{m,n+1}$ for all $m,n \in \mathbb{N}$. The generalized difference double notation has the following binomial representation:

$$\Delta^i x_{m,n} = \sum_{j=0}^{m} \sum_{j=0}^{n} (-1)^{i+j} \binom{m}{i} \binom{n}{j} x_{m+i,j}.$$

2.1 Definition.

Let $A = \left( a_{m,n} \right)$ denote a four dimensional summability method that maps the complex double sequences $x$ into the double sequence $Ax$ where the $k, \ell-th$ term of $Ax$ is as follows:

$$(Ax)_{k,\ell} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n} x_{m,n}$$

such transformation is said to be non-negative if $a_{m,n}$ is non-negative.

The notion of regularity for two dimensional matrix transformations was presented by Silverman and Toepplitz. Following Silverman and Toepplitz, Robison and Hamilton presented the following four dimensional analog of regularity for double sequences in which both added an additional assumption of boundedness. This assumption was made since a double sequence which is $P-$ convergent is not necessarily bounded.

Let $\lambda$ and $\mu$ be two sequence spaces and $A = \left( a_{m,n} \right)$ be a four dimensional infinite matrix of real numbers $\left( a_{m,n} \right)$, where $m,n,k,\ell \in \mathbb{N}$. Then, we say $A$ defines a matrix mapping from $\lambda$ into $\mu$ and we denote it by writing $A: \lambda \rightarrow \mu$. If for every sequence $x = (x_{m,n}) \in \lambda$ the sequence $Ax = (Ax_{k,\ell})$ is convergent, the $A-$ transform of $x$, is in $\mu$. By $(\lambda : \mu)$, we denote the class of all matrices $A$ such that $A: \lambda \rightarrow \mu$. Thus $A \in (\lambda : \mu)$ if and only if the series converges for each $k, \ell \in \mathbb{N}$. A Cartesian product of $n-$metric spaces is the $p-$norm of the $n-$vector of the norms of the $n-$sub spaces.

A trivial example of $p-$product metric of $n-$metric space is the $p-$norm metric space is $X = \mathbb{R}$ equipped with the following metric in the product space is the $p-$norm:

$$\| (d_1(x_1,0), \ldots, d_n(x_n,0)) \|_p = \sup \{ \det (d_{m,n} (x_{m,n}, 0)) \} = \begin{pmatrix} d_1(x_{1,0}) & d_1(x_{1,0}) & \cdots & d_1(x_{1,0}) \\ d_2(x_{2,0}) & \cdots & \cdots & d_2(x_{2,0}) \\ \vdots & \ddots & \ddots & \vdots \\ d_n(x_{n,0}) & \cdots & \cdots & d_n(x_{n,0}) \end{pmatrix}$$

$$\sup \{ \det (d_{m,n} (x_{m,n}, 0)) \}$$

where $x_i = (x_{i,1}, \ldots, x_{i,n}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$.
sequence $x$ is said to be $A$–summable to $\alpha$ if $Ax$ converges to $\alpha$ which is called as the $A$–limit of $x$.

**2.2 Lemma. [See Maddox (1986)]**

Matrix $A = \left( a_{n\ell}' \right)$ is regular if and only if the following three conditions hold:

1. There exists $M > 0$ such that for every $k, \ell = 1, 2, \ldots$
   
   the following inequality holds: $\sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} |a_{n\ell}'| \leq M$

2. $\lim_{k,\ell \to \infty} a_{n\ell}' = 0$ for every $k, \ell = 1, 2, \ldots$

3. $\lim_{k,\ell \to \infty} \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} a_{n\ell}' = 1$.

Let $(q_{n\ell})$ be a sequence of positive numbers and

$$Q_{k\ell} = \sum_{n=0}^{\infty} q_{n\ell} (k, \ell \in \mathbb{N}).$$

Then, the matrix $R^q = \left( r_{n\ell}^q \right)$ of the Riesz mean is given by

$$\left( r_{n\ell}^q \right)^q = \begin{cases} \frac{q_{n\ell}}{Q_{k\ell}} & \text{if } 0 \leq m, n \leq k, \ell \\ 0 & \text{if } (m, n) > k\ell \end{cases} \quad (2.2)$$

The Fibonacci numbers are the sequence of numbers $f_{n\ell}^m \ (k, \ell, m, n \in \mathbb{N})$ defined by the linear recurrence equations $f_{0\ell} = 1$ and $f_{1\ell} = 1, f_{m\ell} = f_{m-1\ell} + f_{m-2\ell}; m, n \geq 2$. Fibonacci numbers have many interesting properties and applications in arts, sciences and architecture. Also, some basic properties of Fibonacci numbers are the following.

$$\sum_{k=1}^{m} \sum_{\ell=1}^{n} f_{m\ell} = f_{m+2\ell} - 1; m, n \geq 1,$$

$$\sum_{k=1}^{m} \sum_{\ell=1}^{n} f_{m\ell}^2 = f_{m+1\ell} f_{m\ell+1}; m, n \geq 1,$$

$$\sum_{k=1}^{m} \sum_{\ell=1}^{n} \frac{1}{f_{m\ell}^m} \text{ converges.}$$

In this paper, we define the Fibonacci matrix $F = \left( f_{m\ell}^m \right)_{m,n=1}^{\infty}$, which differs from existing Fibonacci matrix by using Fibonacci numbers $f_{k\ell}$ and introduce some new sequence spaces $\chi^2$ and $\Lambda^2$. Now, we define the Fibonacci matrix $F = \left( f_{m\ell}^m \right)_{m,n=1}^{\infty}$, by

$$\left( f_{m\ell}^m \right) = \begin{cases} f_{k\ell} & \text{if } 0 \leq k \leq m, 0 \leq \ell \leq n \\ f_{(k+2)(\ell+2)} - 1 & \text{if } (m, n) > k\ell \\ 0 \\ \end{cases}$$

that is,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 2 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 2 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$ 

It is obvious that the four-dimensional infinite matrix $F$ is triangular matrix. Also it follows from lemma 2.2 that the method $F$ is regular.

Let $M$ be an Musielak modulus function. We introduce the following sequence spaces based on the four dimensional infinite matrix $F$:

$$\left[ \Lambda^2_M, \left\| (d(x,0), d(x,0), \ldots, d(x,0)) \right\| \right] = F_{\mu}(x)$$

$$= \sup_{\mu} \left\{ \sum_{m,n=1}^{\infty} M\left( f_{m\ell}^m |x_{m\ell}|^{\mu}, \left\| (d(x,0), d(x,0), \ldots, d(x,0)) \right\| \right) \right\} < \infty.$$

Consider the metric space

$$\left[ \chi^2_M, \left\| (d(x,0), d(x,0), \ldots, d(x,0)) \right\| \right]$$

with the metric

$$d(x, y) = \sup_{m,n \in \mathbb{N}} \left\{ M\left( F_{\mu}(x) - F_{\mu}(y) \right) : m, n = 1, 2, 3, \ldots \right\}. \quad (2.3)$$

Consider the metric space

$$\left[ \chi^2_M, \left\| (d(x,0), d(x,0), \ldots, d(x,0)) \right\| \right]$$

with the metric

$$d(x, y) = \sup_{m,n \in \mathbb{N}} \left\{ M\left( F_{\mu}(x) - F_{\mu}(y) \right) : m, n = 1, 2, 3, \ldots \right\}. \quad (2.4)$$
3. Main Results

3.1 Theorem.

The spaces \( \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] \) and \( \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] \) are BK spaces with the metric (2.4) and (2.5).

**Proof:** By Theorem 4.3.12 of Wilansky (1984) and since the four dimensional infinite matrix \( F \) is triangular, we have the result.

3.2 Theorem.

The spaces \( \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] \) and \( \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] \) are isomorphic to the spaces \( \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] \) and \( \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] \), respectively.

(i.e) \( \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] \) is triangular, it has a unique inverse, and

\[ \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] \approx \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] \]

**Proof:** Let us consider the space of \( \mathcal{X}^2 \), since the four dimensional infinite matrix \( F \) is triangular, it has a unique inverse, which is also triangular [see Tripathy et al. (2009)]. Therefore the linear operator

\[ L_f : \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] \rightarrow \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] \]

\[ L_f (x) = F(x) \text{ for all } x \in \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] \approx \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] , \]

is bijective and is metric preserving by (2.5) in Theorem 3.1. Hence

\[ L_f : \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] \approx \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right]. \]

Similarly the proof for the other space can be established.

3.3 Theorem.

The inclusion

\[ \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] \subset \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] \]

and

\[ \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] \subset \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] \]

holds.

**Proof:** As \( F \) is a regular four dimensional infinite matrix, so the inclusion

\[ \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] \subset \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] \]

is obvious.

Now, let \( x = (x_m) \in \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] \).

Then there is a constant \( M > 0 \) such that \( \|x_m\|^\ell \leq M \) for all \( m,n \in \mathbb{N} \). Thus for each \( k, \ell \in \mathbb{N} \).

\[ \|F x\|_k \leq \left( \frac{1}{\beta_{k,2(\ell+1)}} \sum_{m=1}^{\infty} \left( \|x_m\|^\ell \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right) \right) \]

\[ \leq \left( \frac{M}{\beta_{k,2(\ell+1)}} \sum_{m=1}^{\infty} \left( \|x_m\|^\ell \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right) \right) \]

\[ < \infty \text{ which shows that } F x \in \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right]. \]

Thus we conclude that \( \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] \subset \mathcal{X}^2_{\mathcal{M}_n} \left[ \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right] \).

**Example:** Consider the sequence \( x = (x_m) = \)

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}
\]

Then we have for every \( k, \ell \in \mathbb{N} \), \( F_{\mu} (x) = \frac{1}{\beta_{k,2(\ell+1)}} \sum_{n=1}^{\infty} \sum_{k=1}^{\ell} \left( \|x_m\|^\ell \|d(x_0,0),d(x_2,0),\cdots,d(x_{n-1},0)\|_p \right) \neq 0. \)

This shows that \( F X \in \mathcal{X}^2 \) but \( x \) is not in \( \mathcal{X}^2 \). Thus the
sequence $x$ is in \[ \chi_{M}^{u} \left\| (d(x_1,0), d(x_2,0), \cdots, d(x_n,0)) \right\|_p. \]

Hence the inclusion \[ \chi_{M}^{u} \left\| (d(x_1,0), d(x_2,0), \cdots, d(x_n,0)) \right\|_p \]
\[ \subseteq \chi_{M}^{u} \left\| (d(x_1,0), d(x_2,0), \cdots, d(x_n,0)) \right\|_p \]
\[ \text{is strictly holds.} \]

3.4 Theorem.

The sequence $x = (x_m)$ is not in the set \[ \chi_{M}^{u} \left\| (d(x_1,0), d(x_2,0), \cdots, d(x_n,0)) \right\|_p \]
but in \[ \Lambda_{M}^{2F} \left\| (d(x_1,0), d(x_2,0), \cdots, d(x_n,0)) \right\|_p \]

Proof: Consider the sequence $x = (x_m)$
\[ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \text{ for all } k, \ell \in \mathbb{N}. \]
Then we have for every \[ k, \ell \in \mathbb{N}, F_{n}(x) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} . \]

This shows that \[ F \notin \chi_{M}^{u} \left\| (d(x_1,0), d(x_2,0), \cdots, d(x_n,0)) \right\|_p. \]

Again, consider the sequence $x = (x_m) = \frac{(-1)^m (f_{0m+2k+2m+2\ell} + \cdots)}{f_{nm}}$.

for all $k, \ell \in \mathbb{N}$. Then we have for every $k, \ell \in \mathbb{N}, F_{n}(x) = \begin{pmatrix} (-1)^{nm} & (-1)^{nm} \\ (-1)^{nm} & (-1)^{nm} \\ (-1)^{nm} & (-1)^{nm} \\ \vdots & \vdots \end{pmatrix} . \]

This shows that \[ F \notin \Lambda_{M}^{2F} \left\| (d(x_1,0), d(x_2,0), \cdots, d(x_n,0)) \right\|_p. \]

Competing Interests

Author has declared that no competing interests exist.

Acknowledgement

The authors thank the referee for his careful reading of the manuscript and comments that improved the presentation of the paper.

References


Gökhan, A. and Colak, R. 2004. The double sequence spaces $c_{0}^{2}(p)$ and $c_{0}^{\infty}(p)$, Applied Mathematics Computation. 157(2), 491-501.


Subramanian, N. and Misra, U.K. 2010. The semi normed space defined by a double gai sequence of modulus function, Fasciculi Mathematici. 46.


Zeltser, M. 2001., Investigation of Double Sequence Spaces by Soft and Hard Analytical Methods, Dissertationes Mathematicae Universitatis Tartuensis 25, Tartu University Press, University of Tartu, Faculty of Mathematics and Computer Science, Tartu.