Original Article

Asymptotic stability of time-varying delay-difference system of cellular neural networks via matrix inequalities and application

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Abstract

In this paper, we obtain some criteria for determining the asymptotic stability of the zero solution of time-varying delay-difference system of cellular neural networks in terms of certain matrix inequalities by using a discrete version of the Lyapunov second method. The result is applied to obtain new stability conditions for some classes of time-varying delay-difference equations such as time-varying delay-difference system of cellular neural networks with multiple delays in terms of certain matrix inequalities. Our results can be well suited for computational purposes.

Keywords: asymptotic stability, cellular neural networks, Lyapunov function, time-varying delay-difference system, matrix inequalities

1. Introduction

In recent decades, cellular neural networks have been extensively studied in many aspects and successfully applied to many fields such as pattern identifying, voice recognizing, system controlling, signal processing systems, static image treatment, and solving nonlinear algebraic equations, etc. Such applications are based on the existence of equilibrium points, and qualitative properties of systems. In electronic implementation, time delays occur due to various reasons such as circuit integration, switching delays of the amplifiers and communication delays, etc. Therefore, the study of the asymptotic stability of cellular neural networks with delays is of particular importance to manufacturing high quality microelectronic cellular neural networks.

While stability analysis of continuous-time neural networks can employ the stability theory of differential equations by Wei et al. (2005), it is much harder to study the stability of discrete-time neural networks by Gubta and Jin (1996) with time delays by Arik (2005) or impulses by Liu et al. (2003). The techniques currently available in the literature for discrete-time systems are mostly based on the construction Lyapunov second method by Infante (1978). For Lyapunov second method, it is well known that no general rule exists to guide the construction of a proper Lyapunov function for a given system. In fact, the construction of the Lyapunov function becomes a very difficult task.

In this paper, we consider time varying delay-difference system of cellular neural networks of the form

\[ u(k+1) = -C(k)u(k) + A(k)S(u(k)) + B(k)S(u(k-h)) + f, \]

(1)

where \( u(k) \in \Omega \subseteq \mathbb{R}^n \) is the neuron state vector, \( h \geq 0, \)
\( C(k) = \text{diag}\{c_1(k), ..., c_n(k)\}, \quad c_i(k) \geq 0, \quad i = 1, 2, ..., n \)
is the \( n \times n \) relaxation matrix function, \( A(k) = (a_{ij}(k))_{n \times n} \)
and \( B(k) = (b_{ij}(k))_{n \times n} \) are the \( n \times n \) weight matrix functions,
\( f = (f_1, ..., f_n) \in \mathbb{R}^n \) is the constant external input vector and
\( S(z) = [s_1(z_1), ..., s_n(z_n)]' \) with \( s_i \in C^1[\mathbb{R}, (-1, 1)] \)
where \( s_i \) are the neuron activations and monotonically increasing for each \( i = 1, 2, ..., n \).

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The asymptotic stability of the zero solution of the delay-differential system of cellular neural networks has been developed during the past several years. We refer to monographs by Arik (2002) and Chua and Yang (1988) and the references cited therein. Much less is known regarding the asymptotic stability of the time-varying delay-difference system of cellular neural networks. Therefore, the purpose of this paper is to establish sufficient conditions for the asymptotic stability of the zero solution of (1) in terms of certain matrix inequalities.

2. Preliminaries

The following notations will be used throughout the paper. $\mathbb{R}^+$ denotes the set of all non-negative real numbers; $\mathbb{Z}^+$ denotes the set of all non-negative integers; $\mathbb{R}^n$ denotes the n-finite-dimensional Euclidean space with the Euclidean norm $\|\cdot\|$ and the scalar product between $x$ and $y$ is defined by $x^T y$; $M^{m\times n}$ denotes the space of all $(n \times m)$-matrices; and $A^T$ denotes the transpose of the matrix $A$.

We assume that the neuron activation functions are bounded and satisfy the following hypotheses, respectively:

$$0 \leq \frac{s_i(r_1) - s_i(r_2)}{r_1 - r_2} \leq l_i, \quad \forall r_1, r_2 \in \mathbb{R}, \quad r_1 \neq r_2, \quad (2)$$

where $l_i > 0$ are constants for $i = 1, 2, ..., n$.

By assumption (2) we know that the functions $s_i(\cdot)$ satisfy

$$|s_i(x)| \leq l_i |x|, \quad i = 1, 2, ..., n,$$

and

$$s_i'(x) \leq l_i x s_i(x), \quad (3)$$

Matrix $Q \in \mathbb{R}^{n \times n}$ is positive semidefinite $(Q \succeq 0)$ if $x^T Q x \geq 0$, for all $x \in \mathbb{R}^n$. If $x^T Q x > 0$ for $x \neq 0$, resp., for any $\alpha \neq 0$, then $Q$ is positive (negative, resp.) definite and denoted by $Q > 0, (Q < 0, \text{ resp.})$. It is easy to verify that $Q > 0, (Q < 0, \text{ resp.})$ iff

$$\exists \beta > 0: x^T Q x \geq \beta \|x\|^2, \quad \forall x \in \mathbb{R}^n,$$

$$\exists \beta > 0: x^T Q x \leq -\beta \|x\|^2, \quad \forall x \in \mathbb{R}^n, \quad \text{resp.};$$

Matrix function $Q(t) \in M^{n \times n}$ is positive definite if

$$\exists \beta > 0: x^T Q(t) x \geq \beta \|x\|^2, \quad \forall t \in \mathbb{R}^+, \quad x \in \mathbb{R}^n.$$

Fact 2.1 For any positive scalar $\epsilon$ and vectors $x$ and $y$, the following inequality holds:

$$x^T y + y^T x \leq \epsilon x^T x + \epsilon^{-1} y^T y.$$

Let us denote $V_\delta = \{x \in \mathbb{R}^n : \|x\| < \delta\}$.
Proof Consider the Lyapunov function $V(k, y(k)) = V_1(k, y(k)) + V_2(k, y(k)) + V_3(k, y(k))$ where

$$V_1(k, y(k)) = x^T(k)P(k)x(k),$$

$$V_2(k, y(k)) = \sum_{i=k-\delta+1}^k (h - k + i)x^T(i)G(k)x(i),$$

$$V_3(k, y(k)) = \sum_{i=k-\delta+1}^k x^T(i)W(k)x(i),$$

then difference of $V(k, y(k))$ along trajectory of solution of (4) is given by $\Delta V(k, y(k)) = \Delta V_1(k, y(k)) + \Delta V_2(k, y(k)) + \Delta V_3(k, y(k))$, where

$$\Delta V_1(k, y(k)) = V_1(k, x(k+1)) - V_1(k, x(k))$$

$$= [-C(k)x(k) + A(k)S(x(k)) + B(k)S(x(k-h))]^T P(k+1) x(k) + x(k)[-C(k)x(k) + A(k)S(x(k))] + B(k)S(x(k-h))] - x^T(k)P(k)x(k)$$

$$= x^T(k)C^T(k)P(k+1)C(k) - P(k)x(k) - x^T(k)C^T(k)P(k+1)A(k)S(x(k)) - S^T(x(h))A^T(k)P(k+1)C(k)x(k)$$

$$-x^T(k)C^T(k)P(k+1)B(k)S(x(k-h)) - S^T(x(h))B^T(k)P(k+1)C(k)x(k) - x^T(k)B^T(k)P(k+1)A(k)S(x(k))$$

$$-S^T(x(h))A^T(k)P(k+1)A(k)S(x(k)) + S^T(x(h))A^T(k)P(k+1)B(k)S(x(k-h)) + S^T(x(h))B^T(k)P(k+1)A(k)S(x(k))$$

$$+ S^T(x(h))B^T(k)P(k+1)B(k)S(x(k-h))\)$$

$$\Delta V_2(k, y(k)) = \Delta \left( \sum_{i=k-\delta+1}^k (h - k + i)x^T(i)G(k)x(i) \right)$$

$$= \sum_{i=k-\delta+1}^k (h - k + i)x^T(i)G(k)x(i),$$

and

$$\Delta V_3(k, y(k)) = \Delta \left( \sum_{i=k-\delta+1}^k x^T(i)W(k)x(i) \right) = x^T(k)W(k)x(k) - x^T(k-h)W(k)x(k-h),$$

where (3) and Fact 2.1 are utilized in (6), respectively.

Note that

$$-x^T(k)C^T(k)P(k+1)A(k)S(x(k)) - S^T(x(h))A^T(k)P(k+1)C(k)x(k) \leq$$

$$\varepsilon x^T(k)C^T(k)P(k+1)A(k)A^T(k)P(k+1)C(k)x(k) + \varepsilon^{-1}S^T(x(h))S(x(k)), $$

$$-x^T(k)C^T(k)P(k+1)B(k)S(x(k-h)) - S^T(x(h))B^T(k)P(k+1)C(k)x(k) \leq$$

$$\varepsilon x^T(k)C^T(k)P(k+1)B(k)B^T(k)P(k+1)C(k)x(k) + \varepsilon^{-1}S^T(x(h-h))S(x(k-h)),$$

$$S^T(x(h))A^T(k)P(k+1)A(k)S(x(k)) \leq$$

$$\varepsilon S^T(x(k))A^T(k)P(k+1)B(k)B^T(k)P(k+1)A(k)S(x(k)) + \varepsilon^{-1}S^T(x(h-h))S(x(k-h)),$$

$$e_2S^T(x(h-h))B^T(k)P(k+1)B(k)Lx(k-h) \leq$$

$$x^T(k-h)LB^T(k)P(k+1)B(k)Lx(k-h) + S^T(x(k))A^T(k)P(k+1)A(k)S(x(k)) \leq$$

$$x^T(k-h)LA^T(k)P(k+1)A(k)S(x(k)) + \varepsilon S^T(x(h-h))B^T(k)P(k+1)B(k)Lx(k-h),$$

$$e_2^{-1}S^T(x(h-h))S(x(k-h)) \leq e_2^{-1}x^T(k-h)LLx(k-h)$$

and

$$e_2^{-1}S^T(x(h-h))S(x(k-h)) \leq e_2^{-1}x^T(k-h)LLx(k-h),$$

hence

$$\Delta V_1(k, y(k)) \leq x^T(k)[C^T(k)P(k+1)C(k) - P(k)]x(k) + \varepsilon x^T(k)A^T(k)P(k+1)B(k)B^T(k)P(k+1)A(k)x(k)$$

$$+ e_2 x^T(k)C^T(k)P(k+1)B(k)B^T(k)P(k+1)C(k)x(k) + x^T(k-h)LB^T(k)P(k+1)B(k)Lx(k-h)$$

$$+ x^T(k)LA^T(k)P(k+1)A(k)Lx(k) + e_2 x^T(k)A^T(k)P(k+1)B(k)B^T(k)P(k+1)A(k)S(x(k))$$

$$+ e_2^{-1}x^T(k-h)LLx(k-h) + \varepsilon^{-1}x^T(k)LLx(k),$$

Then we have

$$\Delta V(k, y(k)) \leq x^T(k)[(C^T(k)P(k+1)C(k) - P(k)]x(k) + \varepsilon x^T(k)A^T(k)P(k+1)B(k)B^T(k)P(k+1)A(k)x(k)$$

$$+ e_2 x^T(k)C^T(k)P(k+1)B(k)B^T(k)P(k+1)C(k)x(k) + x^T(k-h)LB^T(k)P(k+1)B(k)Lx(k-h)$$

$$+ x^T(k)LA^T(k)P(k+1)A(k)Lx(k) + e_2 x^T(k)A^T(k)P(k+1)B(k)B^T(k)P(k+1)A(k)S(x(k))$$

$$+ e_2^{-1}x^T(k-h)LLx(k-h) + \varepsilon^{-1}x^T(k)LLx(k).$$
\[ +\varepsilon_i C^T(k) P(k + 1) B(k) B^T(k) P(k + 1) C(k) \\
+\varepsilon_i L A^T(k) P(k + 1) B(k) B^T(k) P(k) A(k) L \\
+ L A^T(k) P(k + 1) A(k) L + \varepsilon_i L L] x(k) \\
+x^T(k - h)[ B(k) B^T(k) P(k + 1) B(k) L + \varepsilon_1 L L + \varepsilon_2 L L \\
-W(k)] x(k - h) - \sum_{i=0}^{\infty} x^T(i) G(k) x(i). \]

Using Lemma 2.2, we obtain:

\[ \sum_{i=0}^{\infty} x^T(i) G(k) x(i) \geq \left( \frac{1}{h} \sum_{i=0}^{\infty} x(i) \right)^T \left( \frac{1}{h} \sum_{i=0}^{\infty} x(i) \right) \]

From the above inequality it follows that:

\[ \Delta V(k, y(k)) \leq \Delta \|
\]

where the matrices are

\[ C(k) = \begin{bmatrix} e^{-2} & 0 \\
0 & e^{-2-0.5}\end{bmatrix} A(k) = \begin{bmatrix} 1.9 - 0.5e^{-3.8i} & -e^{-5.8i} & 1 \\
-e^{-5.8i} & 0.5e^{-2} & -2 \end{bmatrix}, \]

\[ B(k) = \begin{bmatrix} 3 - e^{6i} & -1 \\
0 & -0.5e^{-2i} \end{bmatrix}, s_i(x_i) = \frac{2}{\pi} \tan^{-1}(x_i), i = 1, 2, \]

\[ \varepsilon = 0.5 \text{ and } h = 1. \]

Using the LMI Toolbox in MATLAB, we found that the LMIs in Theorem 3.1 are feasible and

\[ P(k) = \begin{bmatrix} e^{-3.8i} & 0 \\
0 & e^{-0.5} \end{bmatrix}, G(k) = \begin{bmatrix} 1 & 0 \\
0 & 1 \end{bmatrix}, W(k) = \begin{bmatrix} e^{-r} & 0 \\
0 & 1 \end{bmatrix}, \]

and \[ L = \begin{bmatrix} 2.7293 & 0 \\
0 & 2.8806 \end{bmatrix} \] are set of solutions to the LMIs (5).

Therefore, the system is asymptotically stable.

For a given initial condition \[ x(\theta) = [-0.5, -2]^T \], convergence behavior of is shown in Figure 1. As we can see

![Figure 1. The convergence dynamics of the system in example 3.1.](Image 103x230)
4. Application

In this section, we apply the main result to obtain stability conditions for some specific classes of time-varying delay-difference equation. We consider time-varying delay-difference system of cellular neural networks with multiple delays in terms of certain matrix inequalities of the form

$$ u(k+1) = -Cu(k) + AS(u(k)) + \sum_{j=1}^{m} B_j(u(k-h_j)) + f, \quad (7) $$

where $u(k) \in \Omega \subseteq \mathbb{R}^n$ is the neuron state vector, $0 \leq h_j \leq \ldots \leq h_m$, $C(k) = \text{diag}[c_i(k), \ldots, c_k(k)]$, $c_i(k) \geq 0$, $i = 1, 2, \ldots, n$ is the $n \times n$ relaxation matrix function, $A_j(k)$ and $B_j(k)$, $i = 1, 2, \ldots, m$ are the $n \times n$ weight matrices function, $f = (f_1, \ldots, f_m) \in \mathbb{R}^n$ is the constant external input vector and $S(z) = [s_1(z), \ldots, s_n(z)]^T$ with $s_i \in \mathbb{C}^i \mathbb{R}_[-1,1]$ where $s_i$ is the neuron activations and monotonically increasing for each $i = 1, 2, \ldots, n$.

We consider the asymptotic stability of the zero solution $u^*$ of (7) in the terms of certain matrix inequalities. Without loss of generality, we can assume that $u^* = 0$, $S(0) = 0$ and $f(0) = 0$ (for otherwise, we let $x = u - u^*$ and define $S(x) = S(x + u^*) - S(u^*)$).

The new form of (7) is now given by

$$ x(k+1) = -A(k)x(k) + \sum_{j=1}^{m} B_j(k)S(x(k-h_j)). \quad (8) $$

**Theorem 4.1** The zero solution of the time-varying delay-difference system (8) is asymptotically stable if there exist symmetric positive definite matrices $P(k), G_i(k)$, and $W_i(k)$, $i = 1, 2, \ldots, m$ and $L = \text{diag}[l_1, \ldots, l_n] > 0$ satisfying the following matrix inequalities:

$$
W \triangleq \begin{bmatrix}
(0,0) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (0,1) & (1,2) & \cdots & (1,n) & 0 & 0 & 0 & 0 & 0 \\
0 & (2,1) & (2,2) & \cdots & (2,n) & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & (0,1) & (0,2) & \cdots & (0,m) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & (m+1,m+1) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots \\
\end{bmatrix} < 0^+, $$

where

$$ (0,0) = C^T(k)P(k+1)C(k) - P(k) + \sum_{i=1}^{m} h_i G_i(k) + W_i(k) + \varepsilon_1 \sum_{i=1}^{m} A_i^T(k)P(k+1)B_i(k)B_i^T(k)P(k+1)A(k) $$

$$ + \varepsilon_2 \sum_{i=1}^{m} A_i(k)P(k+1)B_i(k)B_i^T(k)P(k+1)A(k) $$

$$ + LA_i(k)P(k+1)A(k) + \varepsilon_3 LL, $$

$$ (1,1) = LB_i(k)P(k+1)B_i(k)L + \varepsilon_1 LL + \varepsilon_2 LL - W_i(k), $$

$$ (1,2) = LB_i(k)P(k+1)B_i(k)L + \varepsilon_1 LL + \varepsilon_2 LL, $$

$$ (1,m) = LB_i(k)P(k+1)B_m(k)L + \varepsilon_1 LL + \varepsilon_2 LL, $$

$$ (2,1) = LB_i(k)P(k+1)B_i(k)L + \varepsilon_1 LL + \varepsilon_2 LL, $$

$$ (2,2) = LB_i(k)P(k+1)B_i(k)L + \varepsilon_1 LL + \varepsilon_2 LL - W_2(k), $$

$$ (2,m) = LB_i(k)P(k+1)B_m(k)L + \varepsilon_1 LL + \varepsilon_2 LL, $$

$$ (m,1) = LB_m(k)P(k+1)B_i(k)L + \varepsilon_1 LL + \varepsilon_2 LL, $$

$$ (m,2) = LB_m(k)P(k+1)B_i(k)L + \varepsilon_1 LL + \varepsilon_2 LL - W_m(k), $$

$$ (m+1,m+1) = -h_i G_i(k), $$

$$ (m+2,m+2) = -h_m G_m(k), $$

and

$$ (2m,2m) = -h_m G_m(k). $$

**Proof** Consider the Lyapunov function $V(k, y(k)) = V_1(k, y(k)) + V_2(k, y(k)) + V_3(k, y(k))$, where

$$ V_1(k, y(k)) = x^T(k)P(k)x(k), $$

$$ V_2(k, y(k)) = \sum_{i=1}^{m} \sum_{j=k-h_i+1}^{k} (h-k+i)x^T(j)G_i(k)x(j), $$

$$ V_3(k, y(k)) = \sum_{i=1}^{m} \sum_{j=k-h_i+1}^{k} x^T(j)W_i(k)x(j), $$

$$ P(k), G_i(k), \text{ and } W_i(k), i = 1, 2, \ldots, m \text{ being symmetric positive definite solutions of (9) and } y(k) = [x(k), x(k-h_1), \ldots, x(k-h_m)]. $$

Then difference of $V(k, y(k))$ along trajectory of solution of (8) is given by

$$ \Delta V_1(k, y(k)) = \Delta V_1(k, y(k)) + \Delta V_2(k, y(k)) + \Delta V_3(k, y(k)), $$

where

$$ \Delta V_1(k, y(k)) = V_1(k, x(k+1)) - V_1(k, x(k)) $$

$$ = [-C(k)x(k) + A(k)S(x(k))x(k) $$

$$ + \sum_{i=1}^{m} B_i(k)S(x(k-h_i))]^T P(k+1) $$

$$ \times (-C(k)x(k) + A(k)S(x(k))x(k)) $$
\[ + \sum_{j=1}^{m} B_j(k)S(x(k-h_j)) - x^T(k)P(k)x(k) \]
\[ = x^T(k)[C^T(k)P(k+1)C(k) - P(k)]x(k) \]
\[ - x^T(k)C^T(k)P(k+1)A(k)S(x(k)) \]
\[ - S^T(x(k))A^T(k)P(k+1)C(k)x(k) \]
\[ - \sum_{j=1}^{m} x^T(k)C^T(k)P(k+1)B_j(k)S(x(k-h_j)) \]
\[ - \sum_{j=1}^{m} S^T(x(k))A^T(k)P(k+1)B_j(k)S(x(k-h_j)) \]
\[ + \sum_{j=1}^{m} S^T(x(k))A^T(k)P(k+1)B_j(k)S(x(k-h_j)) \]
\[ + \sum_{j=1}^{m} S^T(x(k-h_j))B_j^T(k)P(k+1)A(k)S(x(k)) \]
\[ + \sum_{j=1}^{m} S^T(x(k-h_j))B_j^T(k)P(k+1)A(k)S(x(k)) \]
\[ + \sum_{j=1}^{m} S^T(x(k))A^T(k)P(k+1)A(k)S(x(k)) \]
\[ + \sum_{j=1}^{m} \sum_{j=1}^{m} S^T(x(k-h_j))B_j^T(k)P(k+1)B_j(k)S(x(k-h_j)) \]
\[ \Delta V_1(k, y(k)) = \Delta \left( \sum_{j=1}^{m} \sum_{j\neq k-n_j} (h_k - k + j)x^T(j)G_j(k)x(j) \right) \]
\[ = \sum_{j=1}^{m} h_j x^T(k)G_j(k)x(k) - \sum_{j=1}^{m} \sum_{j=1}^{m} x^T(j)G_j(k)x(j), \]
\[ \Delta V_2(k, y(k)) = \Delta \left( \sum_{j=1}^{m} \sum_{j\neq k-n_j} x^T(j)W_j(k)x(j) \right) \]
\[ = \sum_{j=1}^{m} x^T(k)W_j(k)x(k) - \sum_{j=1}^{m} x^T(k-h_j)W_j(k)x(k-h_j). \]

The rest of the proof is similar to that of Theorem 3.1.

5. Conclusions

In this paper, based on a discrete analog of the Lyapunov second method, we have established a sufficient condition for the asymptotic stability of time-varying delay-difference system of cellular neural networks in terms of certain matrix inequalities. The result has been applied to obtain new stability conditions for some classes of time-varying delay-difference equation such as delay-difference system of cellular neural networks with multiple delays in the terms of certain matrix inequalities.

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