On bi-$\Gamma$-ideals in $\Gamma$-semigroups

Ronnason Chinram$^1$ and Chutiporn Jirojkul$^2$

Abstract

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In 1952, R. A. Good and D. R. Hughes introduced the notion of bi-ideals of semigroups and in 1981, the concept of $\Gamma$-semigroups was introduced by M. K. Sen. We have known that $\Gamma$-semigroups are a generalization of semigroups. In this research, the notion of bi-$\Gamma$-ideals in $\Gamma$-semigroups is introduced. We show that bi-$\Gamma$-ideals in $\Gamma$-semigroups are a generalization of bi-ideals in semigroups and we give some properties for bi-$\Gamma$-ideals in $\Gamma$-semigroups. We give the two definitions as follows: A $\Gamma$-semigroup $M$ is called a bi-simple $\Gamma$-semigroup if $M$ is the unique bi-$\Gamma$-ideal of $M$ and a bi-$\Gamma$-ideal $B$ of $M$ is called a minimal bi-$\Gamma$-ideal of $M$ if $B$ does not properly contain any bi-$\Gamma$-ideal of $M$. We show that a bi-$\Gamma$-ideal $B$ of a $\Gamma$-semigroup $M$ is a minimal bi-$\Gamma$-ideal of $M$ if and only if $B$ is a bi-simple $\Gamma$-semigroup.

Key words: bi-$\Gamma$-ideals, $\Gamma$-semigroups, bi-simple $\Gamma$-semigroups, minimal bi-$\Gamma$-ideals

$^1$Ph.D.(Mathematics)  $^2$M.Sc.(Mathematics), Department of Mathematics, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla, 90112
Corresponding e-mail : ronnason.c@psu.ac.th
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Preliminaries

In 1952, R. A. Good and D. R. Hughes have introduced the notion of bi-ideals of semigroups (Good and Hughes, 1952). The first author has studied some properties of bi-ideals in semigroups (Chinram, 2005). Let $S$ be a semigroup. A subsemigroup $B$ of $S$ is called a bi-ideal of $S$ if $BSB \subseteq B$.

**Example 1.1.** Let $S = \{0,1\}$. Then $S$ is a semigroup under the usual multiplication. Let $B = \{0, \frac{1}{2}\}$. Then $B$ is a subsemigroup of $S$. We have that $BSB = \{0, \frac{1}{4}\} \subseteq B$. Therefore $B$ is a bi-ideal of $S$.

**Example 1.2.** Let $N$ be the set of all positive integers. Then $N$ is a semigroup under the usual multiplication. Let $B = 2N$. Thus $BNB = 4N \subseteq 2N = B$. Hence $B$ is a bi-ideal of $N$.

In 1981, the concept of $\Gamma$-semigroups was introduced by M. K. Sen. Let $M$ and $\Gamma$ be any two nonempty sets. If there exists a mapping $M \times \Gamma \times M \rightarrow M$, the image of $(a, \gamma, b)$ by $a \gamma b$, $M$ is called a $\Gamma$-semigroup if $M$ satisfies the identities $(a \gamma b) \mu \omega = a (\gamma (b \mu \omega))$ for all $a, b, c \in M$ and $\gamma, \mu, \omega \in \Gamma$ (Sen, 1981, Sen and Saha, 1986, Saha, 1987). Let $K$ be a nonempty subset of $M$. $K$ is called a sub $\Gamma$-semigroup of $M$ if $a \gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$.

**Example 1.3.** Let $M = \{0,1\}$ and $\Gamma = \{\frac{1}{n} | n \text{ is a positive integer}\}$. Then $M$ is a $\Gamma$-semigroup under the usual multiplication. Next, let $K = \{0, \frac{1}{2}\}$. We have that $K$ is a nonempty subset of $M$ and $a \gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$. Then $K$ is a sub $\Gamma$-semigroup of $M$.

**Example 1.4.** Let $S$ be a semigroup and $\Gamma = \{1\}$. Define a mapping $S \times \Gamma \times S \rightarrow S$ by $a \gamma b = ab$ for all $a, b \in S$. Then $S$ is a $\Gamma$-semigroup.

From Example 1.4, we have seen that every semigroup is a $\Gamma$-semigroup where $\Gamma = \{1\}$. Then $\Gamma$-semigroups are a generalization of semigroups.

In this research, we generalize bi-ideals of semigroups to bi-$\Gamma$-ideals in $\Gamma$-semigroups.

Main results

Let $M$ be a $\Gamma$-semigroup. A sub $\Gamma$-semigroup $B$ of $M$ is called a bi-$\Gamma$-ideal of $M$ if $B \Gamma M \Gamma B \subseteq B$.

**Example 2.1.** Let $S$ be a semigroup, and $\Gamma = \{1\}$. Define a mapping $S \times \Gamma \times S \rightarrow S$ by $a \gamma b = ab$ for all $a, b \in S$. From Example 1.4, we have known that $S$ is a $\Gamma$-semigroup. Let $B$ be a bi-ideal of a semigroup $S$. Thus $BSB \subseteq B$. Since $\Gamma = \{1\}$, $B \Gamma S \Gamma B = BSB \subseteq B$. Hence $B$ is a bi-$\Gamma$-ideal of $S$. 

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Example 2.1 implies that bi-$\Gamma$-ideals in $\Gamma$-semigroups are a generalization of bi-ideals in semigroups (for a suitable $\Gamma$).

**Theorem 2.1.** Let $M$ be a $\Gamma$-semigroup and $B_i$ a bi-$\Gamma$-ideal of $M$ for all $i \in I$. If $\bigcap_{i \in I} B_i \neq \emptyset$, then $\bigcap_{i \in I} B_i$ is a bi-$\Gamma$-ideal of $M$.

**Proof.** Let $M$ be a $\Gamma$-semigroup and $B_i$ a bi-$\Gamma$-ideal of $M$ for all $i \in I$. Assume that $\bigcap_{i \in I} B_i \neq \emptyset$. Let $a, b \in \bigcap_{i \in I} B_i$, $m \in M$ and $\gamma, \mu \in \Gamma$. Then $a, b \in B_i$ for all $i \in I$. Since $B_i$ is a bi-$\Gamma$-ideal of $M$ for all $i \in I$, $a \gamma b \in B_i$ and $a \gamma m \mu \in B_i$ for all $i \in I$. Therefore $a \gamma b \in \bigcap_{i \in I} B_i$ and $a \gamma m \mu \in \bigcap_{i \in I} B_i$. Hence $\bigcap_{i \in I} B_i$ is a bi-$\Gamma$-ideal of $M$.

In Theorem 2.1, $\bigcap_{i \in I} B_i \neq \emptyset$ is a necessary condition. Let $M = (0, 1)$ and $\Gamma = \{1\}$. Then $M$ is a $\Gamma$-semigroup under the usual multiplication. Let $N$ be the set of all positive integers. For $n \in N$, let $B_n = (0, \frac{1}{n})$. It is easy to prove that $B_n$ is a bi-$\Gamma$-ideal of $M$ for all $n \in N$ but $\bigcap_{n \in N} B_n = \emptyset$.

Let $A$ be a nonempty subset of a $\Gamma$-semigroup $M$. Let $\mathcal{Z} = \{ B / B \text{ is a bi-$\Gamma$-ideal of } M \text{ containing } A \}$. Then $\mathcal{Z} \neq \emptyset$ because $M \in \mathcal{Z}$. Let $(A)_n = \bigcap_{i \in N} B_i$. It is clearly seen that $A \subseteq (A)_n$. By Theorem 2.1, $(A)_n$ is a bi-$\Gamma$-ideal of $M$. Moreover, $(A)_n$ is the smallest bi-$\Gamma$-ideal of $M$ containing $A$. $(A)_n$ is called the bi-$\Gamma$-ideal of $M$ generated by $A$.

**Theorem 2.2.** Let $A$ be a nonempty subset of a $\Gamma$-semigroup $M$. Then

$$(A)_n = A \cup A \Gamma A \cup A \Gamma M \Gamma A.$$

**Proof.** Let $A$ be a nonempty subset of a $\Gamma$-semigroup $M$. Let $B = A \cup A \Gamma A \cup A \Gamma M \Gamma A$. Clearly, $A \subseteq B$. We have that $B \Gamma B = (A \cup A \Gamma A \cup A \Gamma M \Gamma A) \Gamma (A \cup A \Gamma A \cup A \Gamma M \Gamma A) \subseteq A \Gamma A \cup A \Gamma M \Gamma A \subseteq B$. Hence $B$ is a sub $\Gamma$-semigroup of $M$.

Since $M$ is a $\Gamma$-semigroup, all elements in $B \Gamma M \Gamma B = (A \cup A \Gamma A \cup A \Gamma M \Gamma A) \Gamma (A \cup A \Gamma A \cup A \Gamma M \Gamma A)$ are in the form of $a \gamma \eta \mu a_2$ for some $a_1, a_2 \in A$, $\gamma, \mu \in \Gamma$ and $m \in M$. Thus $B \Gamma M \Gamma B \subseteq \Gamma \mu A \cup A \Gamma M \Gamma A \subseteq B$. Therefore $B$ is a bi-$\Gamma$-ideal of $M$.

Let $C$ be any bi-$\Gamma$-ideal of $M$ containing $A$. Since $C$ is a sub-$\Gamma$-semigroup of $M$ and $A \subseteq C$, $A \Gamma A \cup A \Gamma M \Gamma A \subseteq C$. Therefore $B = A \cup A \Gamma A \cup A \Gamma M \Gamma A \subseteq C$.

Hence $B$ is the smallest bi-$\Gamma$-ideal of $M$ containing $A$. Therefore $(A)_n = B = A \cup A \Gamma A \cup A \Gamma M \Gamma A$, as required.

**Example 2.2.** Let $N$ be the set of all positive integers and $\Gamma = \{1\}$. Then $N$ is a $\Gamma$-semigroup under usual addition.

(i) Let $A = \{2\}$. We have that $(A)_n = \{2\} \cup \{9\} \cup \{15, 16, 17, \ldots\}$.

(ii) Let $A = \{3, 4\}$. We have that $(A)_n = \{3, 4\} \cup \{11, 12, 13\} \cup \{17, 18, 19, \ldots\}$.

**Theorem 2.3.** Let $M$ be a $\Gamma$-semigroup. Let $B$ be a bi-$\Gamma$-ideal of $M$ and $A$ a nonempty subset of $M$. Then the following statements are true.

(i) $B \Gamma A$ is a bi-$\Gamma$-ideal of $M$.

(ii) $A \Gamma B$ is a bi-$\Gamma$-ideal of $M$.

**Proof.** (i) We have that $B \Gamma A \Gamma (B \Gamma A) = (B \Gamma A \Gamma B \Gamma A) \Gamma (B \Gamma A) \Gamma (B \Gamma A)$ $= (B \Gamma A \Gamma M \Gamma B \Gamma A) \Gamma (B \Gamma A \Gamma M \Gamma B \Gamma A) \subseteq A \Gamma A \cup A \Gamma M \Gamma A \subseteq B$. Therefore $B \Gamma A$ is a bi-$\Gamma$-ideal of $M$.

(ii) The proof of (ii) is similar to the proof of (i).

**Corollary 2.4.** Let $M$ be a $\Gamma$-semigroup. For a positive integer $n$, let $B_1, B_2, \ldots, B_n$ be bi-$\Gamma$-ideals of $M$. Then $B_1 \Gamma B_2 \Gamma \ldots \Gamma B_n$ is a bi-$\Gamma$-ideal of $M$.

**Proof.** We will prove the corollary by mathematical induction. By Theorem 2.3, $B_1 \Gamma B_2$ is a bi-$\Gamma$-ideal of $M$. Next, let $n$ be any positive integer such that $k < n$ and assume $B_1 \Gamma B_2 \Gamma \ldots \Gamma B_k$ is a bi-$\Gamma$-ideal of $M$. We have that $B_1 \Gamma B_2 \Gamma \ldots \Gamma B_k \Gamma B_{k+1} = (B_1 \Gamma B_2 \Gamma \ldots \Gamma B_k) \Gamma B_{k+1}$ is a bi-$\Gamma$-ideal of $M$ by Theorem 2.3.

Let $M$ be a $\Gamma$-semigroup. $M$ is called a bi-simple $\Gamma$-semigroup if $M$ is the unique bi-$\Gamma$-ideal
of $M$. A bi-$\Gamma$-ideal $B$ of $M$ is called a minimal bi-$\Gamma$-ideal of $M$ if $B$ does not properly contain any bi-$\Gamma$-ideal of $M$.

**Example 2.3.** Let $G$ be a group and $\Gamma = G$. Then $G' = G$ and $gG = G = Gg$ for all $g \in G$. Then $G$ is a $\Gamma$-semigroup under the usual binary operation. It is easy to see that $G$ is the unique bi-$\Gamma$-ideal of $G$. Then $G$ is a bi-simple $\Gamma$-semigroup.

**Theorem 2.5.** Let $M$ be a $\Gamma$-semigroup. Then $M$ is a bi-simple $\Gamma$-semigroup if and only if $M = m\Gamma M \Gamma m$ for all $m \in M$, where $m\Gamma M \Gamma m$ means $\{m\} \Gamma M \Gamma \{m\}$.

**Proof.** Let $M$ be a $\Gamma$-semigroup.
Assume that $M$ is a bi-simple $\Gamma$-semigroup. Let $m \in M$. By Theorem 2.3, $m\Gamma M \Gamma m$ is a bi-$\Gamma$-ideal of $M$. Then $M = m\Gamma M \Gamma m$.
Assume that $M = m\Gamma M \Gamma m$ for all $m \in M$.
Let $B$ be a bi-$\Gamma$-ideal of $M$. Let $b \in B$. By assumption, $M = b\Gamma M \Gamma b \subseteq B \Gamma M \Gamma B \subseteq B$. Hence $M = B$. Therefore $M$ is a bi-simple $\Gamma$-semigroup.

**Theorem 2.6.** Let $M$ be a $\Gamma$-semigroup and $B$ a bi-$\Gamma$-ideal of $M$. Then $B$ is a minimal bi-$\Gamma$-ideal of $M$ if and only if $B$ is a bi-simple $\Gamma$-semigroup.

**Proof.** Let $M$ be a $\Gamma$-semigroup and $B$ a bi-$\Gamma$-ideal of $M$.
Assume that $B$ is a minimal bi-$\Gamma$-ideal of $M$. Let $C$ be a bi-$\Gamma$-ideal of $B$. Then $C \Gamma B \Gamma C \subseteq C$. Since $B$ is a bi-$\Gamma$-ideal of $M$, by Theorem 2.3, $C \Gamma B \Gamma C$ is a bi-$\Gamma$-ideal of $M$. Since $B$ is a minimal bi-$\Gamma$-ideal of $M$ and $C \Gamma B \Gamma C \subseteq B$, $C \Gamma B \Gamma C = B$. Hence $B = C \Gamma B \Gamma C \subseteq C$, this implies $B = C$. Then $B$ is a bi-simple $\Gamma$-semigroup.
Assume that $B$ is a bi-simple $\Gamma$-semigroup. Let $C$ be a bi-$\Gamma$-ideal of $M$ such that $C \subseteq B$. Then $C \Gamma B \Gamma C \subseteq C \Gamma M \Gamma C \subseteq C$. Therefore $C$ is a bi-$\Gamma$-ideal of $B$. Since $B$ is a bi-simple $\Gamma$-semigroup, $C = B$. Hence $B$ is a minimal bi-$\Gamma$-ideal of $M$, as required.

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**References**


