Q-fuzzy sets in UP-algebras

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Q-fuzzy sets in UP-algebras∗

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Abstract

In this paper, we introduce the notions of Q-fuzzy UP-ideals and Q-fuzzy UP-subalgebras of UP-algebras, and their properties are investigated. Relations between a Q-fuzzy UP-ideal (resp. Q-fuzzy UP-subalgebra) and a level subsets of a Q-fuzzy set are investigated, and conditions for a Q-fuzzy set to be a Q-fuzzy UP-ideal (resp. Q-fuzzy UP-subalgebra) of A × B, then either μ is a Q-fuzzy UP-ideal (resp. Q-fuzzy UP-subalgebra) of A or δ is a Q-fuzzy UP-ideal (resp. Q-fuzzy UP-subalgebra) of B.

Mathematics Subject Classification: 03G25

Keywords: UP-algebra, Q-fuzzy UP-ideal, Q-fuzzy UP-subalgebra

1 Introduction and Preliminaries

The concept of a fuzzy subset of a set was first considered by Zadeh (1965). The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere.

The concept of Q fuzzy sets is introduced by many researchers and was extensively investigated in many algebraic structures such as: Jun (2001) introduced the notion of Q-fuzzy subalgebras of BCK/BCI-algebras. Roh et al. (2006) studied intuitionistic Q-fuzzy subalgebras of BCK/BCI-algebras. Mostafa et al. (2012) introduced the notions of Q-ideals and fuzzy Q-ideals in Q-algebras. Sitharselvan et al. (2012), Sithar Selvam et al. (2013) and Selvam et al. (2014) introduced and gave some properties anti Q-fuzzy KU-ideals, anti Q-fuzzy KU-subalgebras and anti Q-fuzzy R-closed KU-ideals of KU-algebras. The notion of anti Q-fuzzy R-closed PS-ideals of PS-algebras is introduced, and related properties are investigated Priya and Ramachandran (2014).

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Iampan (2014) introduced a new algebraic structure, called a UP-algebra. In this paper, we introduce the notions of $Q$-fuzzy UP-ideals and $Q$-fuzzy UP-subalgebras of UP-algebras, and their properties are investigated. Relations between a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) and a level subsets of a $Q$-fuzzy set are investigated, and conditions for a $Q$-fuzzy set to be a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) are provided. Finally, prove that it is not true that if $\mu \cdot \delta$ is a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) of $A \times B$, then either $\mu$ is a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) of $A$ or $\delta$ is a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) of $B$.

Before we begin our study, we will introduce the definition of a UP-algebra.

**Definition 1.1.** (Iampan, 2014) An algebra $A = (A; \cdot, 0)$ of type $(2, 0)$ is called a **UP-algebra** if it satisfies the following axioms: for any $x, y, z \in A$,

(UP-1) $y \cdot z \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$, 

(UP-2) $0 \cdot x = x$, 

(UP-3) $x \cdot 0 = 0$, and 

(UP-4) $x \cdot y = y \cdot x = 0$ implies $x = y$.

In (Iampan, 2014) there is given an example of a UP-algebra.

In what follows, let $A$ and $B$ denote UP-algebras unless otherwise specified. The following proposition is very important for the study of a UP-algebra.

**Proposition 1.2.** (Iampan, 2014) In a UP-algebra $A$, the following properties hold: for any $x, y \in A$,

1. $x \cdot x = 0$,
2. $x \cdot y = 0$ and $y \cdot z = 0$ imply $x \cdot z = 0$,
3. $x \cdot y = 0$ implies $(z \cdot x) \cdot (z \cdot y) = 0$,
4. $x \cdot y = 0$ implies $(y \cdot z) \cdot (x \cdot z) = 0$,
5. $x \cdot (y \cdot x) = 0$,
6. $(y \cdot x) \cdot x = 0$ if and only if $x = y \cdot x$, and
7. $x \cdot (y \cdot y) = 0$.

**Definition 1.3.** (Iampan, 2014) A nonempty subset $B$ of $A$ is called a **UP-ideal** of $A$ if it satisfies the following properties:

1. the constant $0$ of $A$ is in $B$, and
2. for any $x, y, z \in A, x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.

Clearly, $A$ and $\{0\}$ are UP-ideals of $A$. 
Q-fuzzy sets in UP-algebras

Theorem 1.4. (Iampan, 2014) Let A be a UP-algebra and \( \{B_i\}_{i \in I} \) a family of UP-ideals of A. Then \( \bigcap_{i \in I} B_i \) is a UP-ideal of A.

Definition 1.5. (Iampan, 2014) A subset S of A is called a UP-subalgebra of A if the constant 0 of A is in S, and \((S; \cdot, 0)\) itself forms a UP-algebra. Clearly, A and \( \{0\} \) are UP-subalgebras of A.

Proposition 1.6. (Iampan, 2014) A nonempty subset S of a UP-algebra \( A = (A; \cdot, 0) \) is a UP-subalgebra of A if and only if S is closed under the \( \cdot \) multiplication on A.

Theorem 1.7. (Iampan, 2014) Let A be a UP-algebra and \( \{B_i\}_{i \in I} \) a family of UP-subalgebras of A. Then \( \bigcap_{i \in I} B_i \) is a UP-subalgebra of A.

Lemma 1.8. (Somjanta et al., 2015) Let f be a fuzzy set in A. Then the following statements hold: for any \( x, y \in A \),

1. \( 1 - \max\{f(x), f(y)\} = \min\{1 - f(x), 1 - f(y)\} \), and
2. \( 1 - \min\{f(x), f(y)\} = \max\{1 - f(x), 1 - f(y)\} \).

Definition 1.9. (Kim, 2006) A \( Q \)-fuzzy set in a nonempty set X (or a \( Q \)-fuzzy subset of X) is an arbitrary function \( f : X \times Q \to [0, 1] \) where Q is a nonempty set and [0, 1] is the unit segment of the real line.

Definition 1.10. A \( Q \)-fuzzy set f in A is called a \( q \)-fuzzy UP-ideal of A if it satisfies the following properties: for any \( x, y, z \in A \),

1. \( f(0, q) \geq f(x, q) \), and
2. \( f(x \cdot z, q) \geq \min\{f(x \cdot (y \cdot z), q), f(y, q)\} \).

A \( Q \)-fuzzy set f in A is called a \( Q \)-fuzzy UP-ideal of A if it is a \( q \)-fuzzy UP-ideal of A for all \( q \in Q \).

Example 1.11. Let \( A = \{0, 1\} \) be a set with a binary operation \( \cdot \) defined by the following Cayley table:

\[
\begin{array}{cc}
0 & 1 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{array}
\]

Then \((A; \cdot, 0)\) is a UP-algebra. Let \( Q = \{a, b\} \). We define a \( Q \)-fuzzy set f in A as follows:

\[
\begin{array}{cc}
f & a & b \\
0 & 0.3 & 0.2 \\
1 & 0.1 & 0.1
\end{array}
\]

Using this data, we can show that f is a \( Q \)-fuzzy UP-ideal of A.
Example 1.12. Let \( A = \{0, 1\} \) be a set with a binary operation \( \cdot \) defined by the following Cayley table:

\[
\begin{array}{c|cc}
\cdot & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{array}
\]

Then \((A; \cdot, 0)\) is a UP-algebra. Let \( Q = \{a, b\} \). We define a \( Q \)-fuzzy set \( f \) in \( A \) as follows:

\[
\begin{array}{c|cc}
f & a & b \\
\hline
0 & 0.3 & 0.1 \\
1 & 0.1 & 0.2 \\
\end{array}
\]

By Example 1.11 we have \( f \) is an \( a \)-fuzzy UP-ideal of \( A \). Since \( f(0, b) = 0.1 < 0.2 = f(1, b) \), we have Definition 1.10(1) is false. Therefore, \( f \) is not a \( b \)-fuzzy UP-ideal of \( A \). Hence, \( f \) is not a \( Q \)-fuzzy UP-ideal of \( A \).

Definition 1.13. A \( Q \)-fuzzy set \( f \) in \( A \) is called a \( q \)-fuzzy UP-subalgebra of \( A \) if for any \( x, y \in A \),

\[
f(x \cdot y, q) \geq \min\{f(x, q), f(y, q)\}.
\]

A \( Q \)-fuzzy set \( f \) in \( A \) is called a \( Q \)-fuzzy UP-subalgebra of \( A \) if it is a \( q \)-fuzzy UP-subalgebra of \( A \) for all \( q \in Q \).

Example 1.14. Let \( A = \{0, 1, 2\} \) be a set with a binary operation \( \cdot \) defined by the following Cayley table:

\[
\begin{array}{c|ccc}
\cdot & 0 & 1 & 2 \\
\hline
0 & 0 & 1 & 2 \\
1 & 0 & 0 & 1 \\
2 & 0 & 0 & 0 \\
\end{array}
\]

Then \((A; \cdot, 0)\) is a UP-algebra. Let \( Q = \{a, b\} \). We defined a \( Q \)-fuzzy set \( f \) in \( A \) as follows:

\[
\begin{array}{c|cc}
f & a & b \\
\hline
0 & 0.4 & 0.7 \\
1 & 0.2 & 0.1 \\
2 & 0.3 & 0.5 \\
\end{array}
\]

Using this data, we can show that \( f \) is a \( Q \)-fuzzy UP-subalgebra of \( A \).

Example 1.15. Let \( A = \{0, 1, 2\} \) be a set with a binary operation \( \cdot \) defined by the following Cayley table:

\[
\begin{array}{c|ccc}
\cdot & 0 & 1 & 2 \\
\hline
0 & 0 & 1 & 2 \\
1 & 0 & 0 & 1 \\
2 & 0 & 0 & 0 \\
\end{array}
\]

Then \((A; \cdot, 0)\) is a UP-algebra. Let \( Q = \{a, b\} \). We defined a \( Q \)-fuzzy set \( f \) in \( A \) as follows:

\[
\begin{array}{c|cc}
f & a & b \\
\hline
0 & 0.4 & 0.1 \\
1 & 0.2 & 0.5 \\
2 & 0.3 & 0.7 \\
\end{array}
\]
By Example [1.14], we have \( f \) is an \( a \)-fuzzy UP-subalgebra of \( A \). Since \( f(1 \cdot 1, b) = 0.1 < 0.5 = \min\{f(1, b), f(1, b)\} \), we have Definition [1.13] is false. Therefore, \( f \) is not a \( b \)-fuzzy UP-subalgebra of \( A \). Hence, \( f \) is not a \( Q \)-fuzzy UP-subalgebra of \( A \).

**Definition 1.16.** (Kim, 2006) Let \( f \) be a \( Q \)-fuzzy set in \( A \). The \( Q \)-fuzzy set \( \overline{f} \) defined by \( \overline{f}(x, q) = 1 - f(x, q) \) for all \( x \in A \) and \( q \in Q \) is called the complement of \( f \) in \( A \).

**Remark 1.17.** For all \( Q \)-fuzzy set \( f \) in \( A \), we have \( f = \overline{f} \).

**Definition 1.18.** Let \( f \) be a \( Q \)-fuzzy set in \( A \). For any \( t \in [0, 1] \), the sets
\[
U(f; t) = \{x \in A \mid f(x, q) \geq t \text{ for all } q \in Q\}
\]
and
\[
U^+(f; t) = \{x \in A \mid f(x, q) > t \text{ for all } q \in Q\}
\]
are called an upper \( t \)-level subset and an upper \( t \)-strong level subset of \( f \), respectively. The sets
\[
L(f; t) = \{x \in A \mid f(x, q) \leq t \text{ for all } q \in Q\}
\]
and
\[
L^-(f; t) = \{x \in A \mid f(x, q) < t \text{ for all } q \in Q\}
\]
are called a lower \( t \)-level subset and a lower \( t \)-strong level subset of \( f \), respectively. For any \( q \in Q \), the sets
\[
U(f; t, q) = \{x \in A \mid f(x, q) \geq t\}
\]
and
\[
U^+(f; t, q) = \{x \in A \mid f(x, q) > t\}
\]
are called a \( q \)-upper \( t \)-level subset and a \( q \)-upper \( t \)-strong level subset of \( f \), respectively. The sets
\[
L(f; t, q) = \{x \in A \mid f(x, q) \leq t\}
\]
and
\[
L^-(f; t, q) = \{x \in A \mid f(x, q) < t\}
\]
are called a \( q \)-lower \( t \)-level subset and a \( q \)-lower \( t \)-strong level subset of \( f \), respectively.

We can easily prove the following two remarks.

**Remark 1.19.** Let \( f \) be a \( Q \)-fuzzy set in \( A \) and for any \( t_1, t_2 \in [0, 1] \) with \( t_1 \leq t_2 \). Then the following properties hold:

1. \( L(f; t_1) \subseteq L(f; t_2) \).
(2) $U(f; t_2) \subseteq U(f; t_1)$,
(3) $L^-(f; t_1) \subseteq L^-(f; t_2)$, and
(4) $U^+(f; t_2) \subseteq U^+(f; t_1)$.

**Remark 1.20.** Let $f$ be a $Q$-fuzzy set in $A$ and for any $t_1, t_2 \in [0, 1]$ with $t_1 \leq t_2$ and $q \in Q$. Then the following properties hold:

(1) $L(f; t_1, q) \subseteq L(f; t_2, q)$,
(2) $U(f; t_2, q) \subseteq U(f; t_1, q)$,
(3) $L^-(f; t_1, q) \subseteq L^-(f; t_2, q)$, and
(4) $U^+(f; t_2, q) \subseteq U^+(f; t_1, q)$.

**Definition 1.21.** [Iampan, 2014] Let $(A; \cdot, 0_A)$ and $(A'; \cdot', 0')$ be UP-algebras. A mapping $f$ from $A$ to $A'$ is called a **UP-homomorphism** if

$$f(x \cdot y) = f(x) \cdot' f(y)$$

for all $x, y \in A$.

A **UP-homomorphism** $f : A \rightarrow A'$ is called a

(1) **UP-endomorphism** of $A$ if $A' = A$,
(2) **UP-epimorphism** if $f$ is surjective,
(3) **UP-monomorphism** if $f$ is injective, and
(4) **UP-isomorphism** if $f$ is bijective. Moreover, we say $A$ is **UP-isomorphic** to $A'$, symbolically, $A \cong A'$, if there is a UP-isomorphism from $A$ to $A'$.

**Proposition 1.22.** [Iampan, 2014] Let $(A; \cdot, 0_A)$ and $(B; *, 0_B)$ be UP-algebras and let $f : A \rightarrow B$ be a UP-homomorphism. Then $f(0_A) = 0_B$.

**Definition 1.23.** [Sithar Selvam et al., 2013] Let $f : A \rightarrow B$ be a function and $\mu$ be a $Q$-fuzzy set in $B$. We define a new $Q$-fuzzy set in $A$ by $\mu_f$ as

$$\mu_f(x, q) = \mu(f(x), q)$$

for all $x \in A$ and $q \in Q$.

**Definition 1.24.** [Sithar Selvam et al., 2013] Let $f : A \rightarrow B$ be a bijection and $\mu_f$ be a $Q$-fuzzy set in $A$. We define a new $Q$-fuzzy set in $B$ by $\mu$ as

$$\mu(y, q) = \mu_f(x, q)$$

where $f(x) = y$ for all $y \in B$ and $q \in Q$.

**Definition 1.25.** [Sithar Selvam et al., 2013] Let $\mu$ be a $Q$-fuzzy set in $A$ and $\delta$ be a $Q$-fuzzy set in $B$. The **Cartesian product** $\mu \times \delta : (A \times B) \rightarrow [0, 1]$ is defined by

$$(\mu \times \delta)((x, y), q) = \max\{\mu(x, q), \delta(y, q)\}$$

for all $x \in A, y \in B$ and $q \in Q$.

The **dot product** $\mu \cdot \delta : (A \times B) \rightarrow [0, 1]$ is defined by

$$(\mu \cdot \delta)((x, y), q) = \min\{\mu(x, q), \delta(y, q)\}$$

for all $x \in A, y \in B$ and $q \in Q$. 
2 Main Results

In this section, we study $Q$-fuzzy UP-ideals and $Q$-fuzzy UP-subalgebras of UP-algebras, and their properties are investigated. Relations between a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) and a level subsets of a $Q$-fuzzy set are investigated, and conditions for a $Q$-fuzzy set to be a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) are provided. Finally, prove that it is not true that if $\mu \cdot \delta$ is a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) of $A \times B$, then either $\mu$ is a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) of $A$ or $\delta$ is a $Q$-fuzzy UP-ideal (resp. $Q$-fuzzy UP-subalgebra) of $B$.

Theorem 2.1. Every $q$-fuzzy UP-ideal of $A$ is a $q$-fuzzy UP-subalgebra of $A$.

Proof. Let $f$ be a $q$-fuzzy UP-ideal of $A$. Let $x, y \in A$. Then

\[
\begin{align*}
    f(x \cdot y, q) &\geq \min\{f(x \cdot (y \cdot y), q), f(y, q)\} & \text{(Definition 1.10 (2))} \\
    &\geq \min\{f(x \cdot 0, q), f(y, q)\} & \text{(Proposition 1.2 (1))} \\
    &\geq \min\{f(y, q)\} & \text{(UP-3)} \\
    &\geq \min\{f(x, q), f(y, q)\} & \text{(Definition 1.10 (1))}
\end{align*}
\]

Hence, $f$ is a $q$-fuzzy UP-subalgebra of $A$.

With Definition 1.10 and Theorem 2.1, we obtain the corollary.

Corollary 2.2. Every $Q$-fuzzy UP-ideal of $A$ is a $Q$-fuzzy UP-subalgebra of $A$.

Theorem 2.3. If $f$ is a $q$-fuzzy UP-subalgebra of $A$, then $f(0, q) \geq f(x, q)$ for all $x \in A$.

Proof. Assume that $f$ is a $q$-fuzzy UP-subalgebra of $A$. By Proposition 1.2 (1) we have $f(0, q) = f(x \cdot x, q) \geq \min\{f(x, q), f(x, q)\} = f(x, q)$ for all $x \in A$.

With Definition 1.13 and Theorem 2.3, we obtain the corollary.

Corollary 2.4. If $f$ is a $Q$-fuzzy UP-subalgebra of $A$, then $f(0, q) \geq f(x, q)$ for all $x \in A$ and $q \in Q$.

We can easily prove the following three lemmas.

Lemma 2.5. Let $f$ be a $Q$-fuzzy set in $A$ and for any $t \in [0, 1]$. Then the following properties hold:

1. $L(f; t) = U(\overline{J}; 1 - t)$,
2. $L^- (f; t) = U^+ (\overline{J}; 1 - t)$,
3. $U(f; t) = L(\overline{J}; 1 - t)$, and
4. $U^+ (f; t) = L^- (\overline{J}; 1 - t)$. 
Lemma 2.6. Let $f$ be a $Q$-fuzzy set in $A$ and for any $t \in [0,1]$ and $q \in Q$. Then the following properties hold:

1. $L(f; t, q) = U(\overline{f}; 1-t, q)$,
2. $L^-(f; t, q) = U^+(\overline{f}; 1-t, q)$,
3. $U(f; t, q) = L(\overline{f}; 1-t, q)$, and
4. $U^+(f; t, q) = L^-(\overline{f}; 1-t, q)$.

Lemma 2.7. Let $f$ be a $Q$-fuzzy set in $A$ and for any $t \in [0,1]$ and $q \in Q$. Then the following properties hold:

1. $L(f; t) = \bigcap_{q\in Q} L(f; t, q)$,
2. $L^-(f; t) = \bigcap_{q\in Q} L^-(f; t, q)$,
3. $U(f; t) = \bigcap_{q\in Q} U(f; t, q)$, and
4. $U^+(f; t) = \bigcap_{q\in Q} U^+(f; t, q)$.

Lemma 2.8. (Malik and Arora, 2014) For any $a, b \in \mathbb{R}$ such that $a < b$, $a < \frac{b-a}{2} < b$.

Theorem 2.9. Let $f$ be a $Q$-fuzzy set in $A$. Then the following statements hold:

1. $\overline{f}$ is a $Q$-fuzzy UP-ideal of $A$ if and only if the following condition $(\ast)$ holds: for any $t \in [0,1]$ and $q \in Q$, $L(f; t, q)$ is either empty or a UP-ideal of $A$,
2. $\overline{f}$ is a $Q$-fuzzy UP-ideal of $A$ if and only if the following condition $(\ast)$ holds: for any $t \in [0,1]$ and $q \in Q$, $L^-(f; t, q)$ is either empty or a UP-ideal of $A$,
3. $f$ is a $Q$-fuzzy UP-ideal of $A$ if and only if the following condition $(\ast)$ holds: for any $t \in [0,1]$ and $q \in Q$, $U(f; t, q)$ is either empty or a UP-ideal of $A$,
4. $f$ is a $Q$-fuzzy UP-ideal of $A$ if and only if the following condition $(\ast)$ holds: for any $t \in [0,1]$ and $q \in Q$, $U^+(f; t, q)$ is either empty or a UP-ideal of $A$.

Proof. (1) Assume that $\overline{f}$ is a $Q$-fuzzy UP-ideal of $A$. Then $\overline{f}$ is a $q$-fuzzy UP-ideal of $A$ for all $q \in Q$. Let $q \in Q$ and $t \in [0,1]$ be such that $L(f; t, q) \neq \emptyset$ and let $x \in L(f; t, q)$. Then $f(x, q) \leq t$. Now,

\[
\overline{f}(0, q) = \overline{f}(x \cdot 0, q) \\
\geq \min\{\overline{f}(x \cdot (x \cdot 0), q), \overline{f}(x, q)\} \\
= \min\{\overline{f}(x \cdot 0, q), \overline{f}(x, q)\} \\
= \min\{\overline{f}(0, q), \overline{f}(x, q)\} \\
= \overline{f}(x, q). 
\]

(UP-3) (Definition 1.10(2)) (UP-3) (UP-3) (Definition 1.10(1))
Then \(1 - f(0, q) \geq 1 - f(x, q)\), so \(f(0, q) \leq f(x, q) \leq t\). Hence, \(0 \in L(f; t, q)\). Let \(x, y, z \in A\) be such that \(x \cdot (y \cdot z) \in L(f; t, q)\) and \(y \in L(f; t, q)\). Then \(f(x \cdot (y \cdot z), q) \leq t\) and \(f(y, q) \leq t\). By Definition \(1.10(2)\) we have \(\overline{f}(x \cdot z, q) \geq \min\{\overline{f}(x \cdot (y \cdot z), q), \overline{f}(y, q)\}\). Thus

\[
1 - f(x \cdot z, q) \geq \min\{1 - f(x \cdot (y \cdot z), q), 1 - f(y, q)\} = 1 - \max\{f(x \cdot (y \cdot z), q), f(y, q)\}.
\]

(1)

Then \(f(x \cdot z, q) \leq \max\{f(x \cdot (y \cdot z), q), f(y, q)\} \leq t\). Hence, \(x \cdot z \in L(f; t, q)\). Therefore, \(L(f; t, q)\) is a UP-ideal of \(A\).

Conversely, assume that the condition (*) holds and suppose that \(\overline{f}(0, q) \geq \overline{f}(x, q)\) for all \(x \in A\) and \(q \in Q\) is false. Then there exist \(x \in A\) and \(q \in Q\) such that \(\overline{f}(0, q) < \overline{f}(x, q)\). Thus \(1 - f(0, q) < 1 - f(x, q)\), so \(f(0, q) > f(x, q)\). Let \(t = \frac{f(0, q) + f(x, q)}{2}\). Then \(t \in [0, 1]\) and by Lemma \(2.8\) we have \(f(0, q) > t > f(x, q)\). Thus \(x \in L(f; t, q)\), so \(L(f; t, q) \neq \emptyset\). By assumption, we have \(L(f; t, q)\) is a UP-ideal of \(A\). It follows that \(0 \in L(f; t, q)\), so \(f(0, q) \leq t\) which is a contradiction.

Hence, \(\overline{f}(0, q) \geq \overline{f}(x, q)\) for all \(x \in A\) and \(q \in Q\). Suppose that \(\overline{f}(x \cdot z, q) \geq \min\{\overline{f}(x \cdot (y \cdot z), q), \overline{f}(y, q)\}\) for all \(x, y, z \in A\) and \(q \in Q\) is false. Then there exist \(x, y, z \in A\) and \(q \in Q\) such that \(\overline{f}(x \cdot z, q) < \min\{\overline{f}(x \cdot (y \cdot z), q), \overline{f}(y, q)\}\). Thus

\[
1 - f(x \cdot z, q) < \min\{1 - f(x \cdot (y \cdot z), q), 1 - f(y, q)\} = 1 - \max\{f(x \cdot (y \cdot z), q), f(y, q)\}.
\]

Then \(f(x \cdot z, q) > \max\{f(x \cdot (y \cdot z), q), f(y, q)\}\). Let \(g_0 = \frac{f(x \cdot (y \cdot z), q) + f(y, q)}{2}\). Then \(g_0 \in [0, 1]\) and by Lemma \(2.8\) we have \(f(x \cdot z, q) > g_0 > \max\{f(x \cdot (y \cdot z), q), f(y, q)\}\). Thus \(f(x \cdot (y \cdot z), q) < g_0\) and \(f(y, q) < g_0\), so \(x \cdot (y \cdot z) \in L(f; g_0, q)\) and \(y \in L(f; g_0, q)\), so \(L(f; g_0, q) \neq \emptyset\). By assumption, we have \(L(f; g_0, q)\) is a UP-ideal of \(A\). It follows that \(x \cdot z \in L(f; g_0, q)\), so \(f(x \cdot z, q) \leq g_0\) which is a contradiction.

Hence, \(\overline{f}(x \cdot z, q) \geq \min\{\overline{f}(x \cdot (y \cdot z), q), \overline{f}(y, q)\}\) for all \(x, y, z \in A\) and \(q \in Q\). Therefore, \(\overline{f}\) is a \(Q\)-fuzzy UP-ideal of \(A\) for all \(q \in Q\). Consequently, \(\overline{f}\) is a \(Q\)-fuzzy UP-ideal of \(A\).

(2) Similarly to as in the proof of (1).

(3) Assume that \(f\) is a \(Q\)-fuzzy UP-ideal of \(A\). Then \(f\) is a \(Q\)-fuzzy UP-ideal of \(A\) for all \(q \in Q\). Let \(q \in Q\) and \(t \in [0, 1]\) be such that \(U(f; t, q) \neq \emptyset\) and let \(x \in U(f; t, q)\). Then \(f(x, q) \geq t\). Now,

\[
f(0, q) = f(x \cdot 0, q) \geq \min\{f(x \cdot (x \cdot 0), q), f(x, q)\} = \min\{f(x \cdot 0, q), f(x, q)\} = \min\{f(0, q), f(x, q)\} = f(x, q) \geq t.
\]

Hence, \(0 \in U(f; t, q)\). Let \(x, y, z \in A\) be such that \(x \cdot (y \cdot z) \in U(f; t, q)\) and \(y \in U(f; t, q)\). Then \(f(x \cdot (y \cdot z), q) \geq t\) and \(f(y, q) \geq t\). By Definition \(1.10(2)\), we
have \( f(x \cdot z, q) \geq \min \{ f(x \cdot (y \cdot z), q), f(y, q) \} \geq t \). Thus \( x \cdot z \in U(f; t, q) \). Hence, \( U(f; t, q) \) is a UP-ideal of \( A \).

Conversely, assume that the condition (*) holds and suppose that \( f(0, q) \geq f(x, q) \) for all \( x \in A \) and \( q \in Q \) is false. Then there exist \( x \in A \) and \( q \in Q \) such that \( f(0, q) < f(x, q) \). Let \( t = \frac{f(0, q) + f(x, q)}{2} \). Then \( t \in [0, 1] \) and by Lemma [2.8] we have \( f(0, q) < t < f(x, q) \). Thus \( x \in U(f; t, q) \), so \( U(f; t, q) \neq \emptyset \). By assumption, we have \( U(f; t, q) \) is a UP-ideal of \( A \). It follows that \( 0 \in U(f; t, q) \), so \( f(0, q) \geq t \) which is a contradiction. Hence, \( f(0, q) \geq f(x, q) \) for all \( x \in A \) and \( q \in Q \). Suppose that \( f(x \cdot z, q) \geq \min \{ f(x \cdot (y \cdot z), q), f(y, q) \} \) for all \( x, y, z \in A \) and \( q \in Q \) is false. Then there exist \( x, y, z \in A \) and \( q \in Q \) such that \( f(x \cdot z, q) < \min \{ f(x \cdot (y \cdot z), q), f(y, q) \} \). Let \( g_0 = \frac{f(x \cdot z, q) + \min \{ f(x \cdot (y \cdot z), q), f(y, q) \}}{2} \). Then \( g_0 \in [0, 1] \) and by Lemma [2.8] we have \( f(x \cdot z, q) < g_0 < \min \{ f(x \cdot (y \cdot z), q), f(y, q) \} \). Thus \( f(x \cdot (y \cdot z), q) > g_0 \) and \( f(y, q) > g_0 \), so \( x \cdot (y \cdot z) \in U(f; g_0, q) \) and \( y \in U(f; g_0, q) \), so \( U(f; g_0, q) \neq \emptyset \). By assumption, we have \( U(f; g_0, q) \) is a UP-ideal of \( A \). It follows that \( x \in U(f; g_0, q) \), so \( f(x \cdot z, q) \geq g_0 \) which is a contradiction. Hence, \( f(x \cdot z, q) \geq \min \{ f(x \cdot (y \cdot z), q), f(y, q) \} \) for all \( x, y, z \in A \) and \( q \in Q \). Therefore, \( f \) is a \( q \)-fuzzy UP-ideal of \( A \) for all \( q \in Q \). Consequently, \( f \) is a \( Q \)-fuzzy UP-ideal of \( A \).

(4) Similarly to as in the proof of (3) \( \square \)

**Corollary 2.10.** Let \( f \) be a \( Q \)-fuzzy set in \( A \). Then the following statements hold:

(1) if \( f \) is a \( Q \)-fuzzy UP-ideal of \( A \), then for any \( t \in [0, 1] \), \( L(f; t) \) is either empty or a UP-ideal of \( A \),

(2) if \( f \) is a \( Q \)-fuzzy UP-ideal of \( A \), then for any \( t \in [0, 1] \), \( L^{-}(f; t) \) is either empty or a UP-ideal of \( A \),

(3) if \( f \) is a \( Q \)-fuzzy UP-ideal of \( A \), then for any \( t \in [0, 1] \), \( U(f; t) \) is either empty or a UP-ideal of \( A \), and

(4) if \( f \) is a \( Q \)-fuzzy UP-ideal of \( A \), then for any \( t \in [0, 1] \), \( U^{+}(f; t) \) is either empty or a UP-ideal of \( A \).

**Proof.** (1) Assume that \( f \) is a \( Q \)-fuzzy UP-ideal of \( A \). By Theorem 2.9 [1], we have that for any \( t \in [0, 1] \) and \( q \in Q \), \( L(f; t, q) \) is either empty or a UP-ideal of \( A \). Let \( t \in [0, 1] \). If \( L(f; t, q) = \emptyset \) for some \( q \in Q \), it follows from Lemma 2.7 [1] that \( L(f; t) = \bigcap_{q \in Q} L(f; t, q) = \emptyset \). If \( L(f; t, q) \neq \emptyset \) for all \( q \in Q \), it follows from Theorem 2.9 [1] that \( L(f; t, q) \) is a UP-ideal of \( A \) for all \( q \in Q \). By Lemma 2.7 [3] and Theorem 1.4, we have \( L(f; t) = \bigcap_{q \in Q} L(f; t, q) \) is a UP-ideal of \( A \).

(2) Similarly to as in the proof of (1) \( \square \)

(3) Assume that \( f \) is a \( Q \)-fuzzy UP-ideal of \( A \). By Theorem 2.9 [3], we have that for any \( t \in [0, 1] \) and \( q \in Q \), \( U(f; t, q) \) is either empty or a UP-ideal of \( A \). Let \( t \in [0, 1] \). If \( U(f; t, q) = \emptyset \) for some \( q \in Q \), it follows from Lemma 2.7 [3] that \( U(f; t) = \bigcap_{q \in Q} U(f; t, q) = \emptyset \). If \( U(f; t, q) \neq \emptyset \) for all \( q \in Q \), it follows from Theorem 2.9 [3] that \( U(f; t, q) \) is a UP-ideal of \( A \) for all \( q \in Q \). By Lemma 2.7 [3] and Theorem 1.4, we have \( U(f; t) = \bigcap_{q \in Q} U(f; t, q) \) is a UP-ideal of \( A \).

(4) Similarly to as in the proof of (3) \( \square \)
Theorem 2.11. Let $f$ be a $Q$-fuzzy set in $A$. Then the following statements hold:

(1) $\overline{T}$ is a $Q$-fuzzy UP-subalgebra of $A$ if and only if the following condition (⋆) holds: for any $t \in [0,1]$ and $q \in Q$, $L(f; t, q)$ is either empty or a UP-subalgebra of $A$,

(2) $\overline{T}$ is a $Q$-fuzzy UP-subalgebra of $A$ if and only if the following condition (⋆) holds: for any $t \in [0,1]$ and $q \in Q$, $L^-(f; t, q)$ is either empty or a UP-subalgebra of $A$,

(3) $f$ is a $Q$-fuzzy UP-subalgebra of $A$ if and only if the following condition (⋆) holds: for any $t \in [0,1]$ and $q \in Q$, $U(f; t, q)$ is either empty or a UP-subalgebra of $A$, and

(4) $f$ is a $Q$-fuzzy UP-subalgebra of $A$ if and only if the following condition (⋆) holds: for any $t \in [0,1]$ and $q \in Q$, $U^+(f; t, q)$ is either empty or a UP-subalgebra of $A$.

Proof. (1) Assume that $\overline{T}$ is a $Q$-fuzzy UP-subalgebra of $A$. Then $\overline{T}$ is a $q$-fuzzy UP-subalgebra of $A$ for all $q \in Q$. Let $q \in Q$ and $t \in [0,1]$ be such that $L(f; t, q) \neq \emptyset$ and let $x, y \in L(f; t, q)$. Then $f(x, q) \leq t$ and $f(y, q) \leq t$. Now,

$$\overline{T}(x \cdot y, q) = \min\{\overline{T}(x, q), \overline{T}(y, q)\} = \min\{1 - f(x, q), 1 - f(y, q)\}$$

$$= 1 - \max\{f(x, q), f(y, q)\}. \quad (\text{Lemma L8(1)})$$

Then $f(x \cdot y, q) \leq \max\{f(x, q), f(y, q)\} \leq t$, so $x \cdot y \in L(f; t, q)$. Hence, $L(f; t, q)$ is a UP-subalgebra of $A$.

Conversely, assume that the condition (⋆) holds. Let $x, y \in A$ and $q \in Q$ and let $t = \max\{f(x, q), f(y, q)\}$. Thus $f(x, q) \leq t$ and $f(y, q) \leq t$, so $x, y \in L(f; t, q)$.

By assumption, we have $L(f; t, q)$ is a UP-subalgebra of $A$. It follows that $x \cdot y \in L(f; t, q)$. Thus $f(x \cdot y, q) \leq t = \max\{f(x, q), f(y, q)\}$, so

$$1 - f(x \cdot y, q) \geq 1 - \max\{f(x, q), f(y, q)\}$$

$$= \min\{1 - f(x, q), 1 - f(y, q)\}. \quad (\text{Lemma L8(1)})$$

Hence, $\overline{T}(x \cdot y, q) \geq \min\{\overline{T}(x, q), \overline{T}(y, q)\}$. Therefore, $\overline{T}$ is a $Q$-fuzzy UP-subalgebra of $A$ for all $q \in Q$. Consequently, $\overline{T}$ is a $Q$-fuzzy UP-subalgebra of $A$.

(2) Similarly to as in the proof of the necessity of [1].

Conversely, assume that the condition (⋆) holds. Assume that there exist $x, y \in A$ and $q \in Q$ such that $\overline{T}(x \cdot y, q) < \min\{\overline{T}(x, q), \overline{T}(y, q)\}$. By Lemma L8(1), we have

$$1 - f(x \cdot y, q) < \min\{1 - f(x, q), 1 - f(y, q)\} = 1 - \max\{f(x, q), f(y, q)\}.$$ 

Thus $f(x \cdot y, q) > \max\{f(x, q), f(y, q)\}$. Now $f(x \cdot y, q) \in [0,1]$, we choose $t = f(x \cdot y, q)$. Thus $f(x, q) < t$ and $f(y, q) < t$, so $x, y \in L^-(f; t, q) \neq \emptyset$. By assumption, we have $L^-(f; t, q)$ is a UP-subalgebra of $A$ and so $x \cdot y \in L^-(f; t, q)$. Thus $f(x \cdot y, q) < t = f(x \cdot y, q)$ which is a contradiction. Hence, $\overline{T}(x \cdot y, q) \geq \min\{\overline{T}(x, q), \overline{T}(y, q)\}$ for all $x, y \in A$ and $q \in Q$. Therefore, $\overline{T}$ is a $Q$-fuzzy UP-subalgebra of $A$ for all $q \in Q$. Consequently, $\overline{T}$ is a $Q$-fuzzy UP-subalgebra of $A$. 
(3) Assume that \( f \) is a \( Q \)-fuzzy UP-subalgebra of \( A \). Then \( f \) is a \( q \)-fuzzy UP-subalgebra of \( A \) for all \( q \in Q \). Let \( q \in Q \) and \( t \in [0, 1] \) be such that \( U(f; t, q) \neq \emptyset \) and let \( x, y \in U(f; t, q) \). Then \( f(x, q) \geq t \) and \( f(y, q) \geq t \), we have \( f(x \cdot y, q) \geq \min \{ f(x, q), f(y, q) \} \geq t \). Thus \( x \cdot y \in U(f; t, q) \). Hence, \( U(f; t, q) \) is a UP-subalgebra of \( A \).

Conversely, assume that the condition \((\ast)\) holds. Let \( x, y \in A \) and \( q \in Q \) and let \( t = \min \{ f(x, q), f(y, q) \} \). Thus \( f(x, q) \geq t \) and \( f(y, q) \geq t \), so \( x, y \in U(f; t, q) \neq \emptyset \). By assumption, we have \( U(f; t, q) \) is a UP-subalgebra of \( A \). It follows that \( x \cdot y \in U(f; t, q) \). Thus \( f(x \cdot y, q) \geq t = \min \{ f(x, q), f(y, q) \} \). Hence, \( f \) is a \( Q \)-fuzzy UP-subalgebra of \( A \) for all \( q \in Q \). Consequently, \( f \) is a \( Q \)-fuzzy UP-subalgebra of \( A \).

(4) Similarly to as in the proof of the necessity of \((3)\).

Conversely, assume that the condition \((\ast)\) holds. Assume that there exist \( x, y \in A \) and \( q \in Q \) such that \( f(x \cdot y, q) < \min \{ f(x, q), f(y, q) \} \). Then \( f(x \cdot y, q) \in [0, 1) \). Choose \( t = f(x \cdot y, q) \). Thus \( f(x, q) > t \) and \( f(y, q) > t \), so \( x, y \in U^{+}(f; t, q) \neq \emptyset \). By assumption, we have \( U^{+}(f; t, q) \) is a UP-subalgebra of \( A \) and so \( x \cdot y \in U^{+}(f; t, q) \).

Thus \( f(x \cdot y, q) > t = f(x, q) \) which is a contradiction. Hence, \( f(x \cdot y, q) \geq \min \{ f(x, q), f(y, q) \} \) for all \( x, y \in A \) and \( q \in Q \). Therefore, \( f \) is a \( Q \)-fuzzy UP-subalgebra of \( A \) for all \( q \in Q \). Consequently, \( f \) is a \( Q \)-fuzzy UP-subalgebra of \( A \).

**Corollary 2.12.** Let \( f \) be a \( Q \)-fuzzy set in \( A \). Then the following statements hold:

1. if \( \overline{f} \) is a \( Q \)-fuzzy UP-subalgebra of \( A \), then for any \( t \in [0, 1] \), \( L(f; t) \) is either empty or a UP-subalgebra of \( A \),
2. if \( \overline{f} \) is a \( Q \)-fuzzy UP-subalgebra of \( A \), then for any \( t \in [0, 1] \), \( L^{-}(f; t) \) is either empty or a UP-subalgebra of \( A \),
3. if \( f \) is a \( Q \)-fuzzy UP-subalgebra of \( A \), then for any \( t \in [0, 1] \), \( U(f; t) \) is either empty or a UP-subalgebra of \( A \),
4. if \( f \) is a \( Q \)-fuzzy UP-subalgebra of \( A \), then for any \( t \in [0, 1] \), \( U^{+}(f; t) \) is either empty or a UP-subalgebra of \( A \).

**Proof.** (1) Assume that \( \overline{f} \) is a \( Q \)-fuzzy UP-subalgebra of \( A \). By Theorem 2.11 \((1)\) we have for any \( t \in [0, 1] \) and \( q \in Q \), \( L(f; t, q) \) is either empty or a UP-subalgebra of \( A \). Let \( t \in [0, 1] \). If \( L(f; t, q) = \emptyset \) for some \( q \in Q \), it follows from Lemma 2.7 \((1)\) that \( L(f; t) = \bigcap_{q \in Q} L(f; t, q) = \emptyset \). If \( L(f; t, q) \neq \emptyset \) for all \( q \in Q \), it follows from Theorem 2.11 \((1)\) that \( L(f; t, q) \) is a UP-subalgebra of \( A \) for all \( q \in Q \). By Lemma 2.7 \((1)\) and Theorem 1.7, we have \( L(f; t) = \bigcap_{q \in Q} L(f; t, q) \) is a UP-subalgebra of \( A \).

(2) Similarly to as in the proof of \((1)\).

(3) Assume that \( f \) is a \( Q \)-fuzzy UP-subalgebra of \( A \). By Theorem 2.11 \((3)\) we have for any \( t \in [0, 1] \) and \( q \in Q \), \( U(f; t, q) \) is either empty or a UP-subalgebra of \( A \). Let \( t \in [0, 1] \). If \( U(f; t, q) = \emptyset \) for some \( q \in Q \), it follows from Lemma 2.7 \((3)\) that \( U(f; t) = \bigcap_{q \in Q} U(f; t, q) = \emptyset \). If \( U(f; t, q) \neq \emptyset \) for all \( q \in Q \), it follows from Theorem 2.11 \((3)\) that \( U(f; t, q) \) is a UP-subalgebra of \( A \) for all \( q \in Q \). By Lemma 2.7 \((3)\) and Theorem 1.7, we have \( U(f; t) = \bigcap_{q \in Q} U(f; t, q) \) is a UP-subalgebra of \( A \).

(4) Similarly to as in the proof of \((3)\) \( \square \)
Corollary 2.13. Let $I$ be a $UP$-ideal of $A$. Then the following statements hold:

(1) for any $k \in (0,1]$, then there exists a $Q$-fuzzy UP-ideal $g$ of $A$ such that $L(g; t) = I$ for all $t < k$ and $L(g; t) = A$ for all $t \geq k$, and

(2) for any $k \in [0,1)$, then there exists a $Q$-fuzzy UP-ideal $f$ of $A$ such that $U(f; t) = I$ for all $t > k$ and $U(f; t) = A$ for all $t \leq k$.

Proof. (1) Let $f$ be a $Q$-fuzzy set in $A$ defined by

$$f(x, q) = \begin{cases} 0 & \text{if } x \in I, \\ k & \text{if } x \notin I, \end{cases}$$

for all $q \in Q$.

Case 1: To show that $L(f; t) = I$ for all $t < k$, let $t \in [0,1]$ be such that $t < k$. Let $x \in L(f; t)$. Then $f(x, q) \leq t < k$ for all $q \in Q$. Thus $f(x, q) \neq k$ for all $q \in Q$, so $f(x, q) = 0$ for all $q \in Q$. Thus $x \in I$, so $L(f; t) \subseteq I$. Now, let $x \in I$. Then $f(x, q) = 0 \leq t$ for all $q \in Q$. Thus $x \in L(f; t)$, so $I \subseteq L(f; t)$. Hence, $L(f; t) = I$ for all $t < k$.

Case 2: To show that $L(f; t) = A$ for all $t \geq k$, let $t \in [0,1]$ be such that $t \geq k$. Clearly, $L(f; t) \subseteq A$. Let $x \in A$. Then

$$f(x, q) = \begin{cases} 0 & \text{if } x \in I, \\ k & \text{if } x \notin I, \end{cases}$$

for all $q \in Q$. Thus $x \in L(f; t)$, so $A \subseteq L(f; t)$. Hence, $L(f; t) = A$ for all $t \geq k$. We claim that $L(f; t, q) = L(f; t, q')$ for all $q, q' \in Q$. For $q, q' \in Q$, we obtain

$$x \in L(f; t, q) \iff f(x, q) \leq t$$

$$\iff f(x, q') \leq t \quad (f(x, q) = f(x, q'))$$

$$\iff x \in L(f; t, q').$$

Hence, $L(f; t, q) = L(f; t, q')$ for all $q, q' \in Q$. By Lemma 2.7 [1], we have $L(f; t) = \bigcap_{q \in Q} L(f; t, q)$. By the claim, we have $L(f; t) = L(f; t, q)$ for all $q \in Q$. Since $L(f; t, q) = L(f; t) = I$ for all $t < k$ and $L(f; t, q) = L(f; t) = A$ for all $t \geq k$, it follows from Theorem 2.9 [1] that $\overline{f}$ is a $Q$-fuzzy UP-ideal of $A$. By Remark 1.17, we have $L(\overline{f}; t) = L(f; t) = I$ for all $t < k$ and $L(\overline{f}; t) = L(f; t) = A$ for all $t \geq k$.

Let $\overline{f} = g$. Then $g$ is a $Q$-fuzzy UP-ideal of $A$ such that $L(g; t) = I$ for all $t < k$ and $L(g; t) = A$ for all $t \geq k$.

(2) Let $f$ be a $Q$-fuzzy set in $A$ defined by

$$f(x, q) = \begin{cases} 1 & \text{if } x \in I, \\ k & \text{if } x \notin I, \end{cases}$$

for all $q \in Q$.

Case 1: To show that $U(f; t) = I$ for all $t > k$, let $t \in [0,1]$ be such that $t > k$. Let $x \in U(f; t)$. Then $f(x, q) \geq t > k$ for all $q \in Q$. Thus $f(x, q) \neq k$ for all $q \in Q$, so $f(x, q) = 1$ for all $q \in Q$. Thus $x \in I$, so $U(f; t) \subseteq I$. Now, let $x \in I$. Then

Case 2: To show that $U(f; t) = A$ for all $t \leq k$, let $t \in [0,1]$ be such that $t \leq k$. Let $x \in U(f; t)$. Then $f(x, q) \leq t < k$ for all $q \in Q$. Thus $f(x, q) = 0$ for all $q \in Q$, so $f(x, q) = 0$ for all $q \in Q$. Thus $x \notin I$, so $U(f; t) \supseteq I$. Now, let $x \notin I$. Then $f(x, q) = 0 \geq t$ for all $q \in Q$. Thus $x \notin U(f; t)$, so $I \supseteq U(f; t)$. Hence, $U(f; t) = A$ for all $t \leq k$.
f(x, q) = 1 \geq t \text{ for all } q \in Q. \text{ Thus } x \in U(f; t), \text{ so } I \subseteq U(f; t). \text{ Hence, } U(f; t) = I \text{ for all } t > k.

Case 2: To show that \( U(f; t) = A \) for all \( t \leq k \), let \( t \in [0, 1] \) be such that \( t \leq k \). Clearly, \( U(f; t) \subseteq A \). Let \( x \in A \). Then

\[
f(x, q) = \begin{cases} 
  k \geq t & \text{if } x \notin I, \\
  1 > t & \text{if } x \in I,
\end{cases}
\]

for all \( q \in Q \). Thus \( x \in U(f; t) \), so \( A \subseteq U(f; t) \). Hence, \( U(f; t) = A \) for all \( t \leq k \).

We claim that \( U(f; t, q) = U(f; t, q') \) for all \( q, q' \in Q \). For \( q, q' \in Q \), we obtain

\[
x \in U(f; t, q) \iff f(x, q) \geq t
\]

\[
\iff \{ f(x, q') \geq t \} \quad (f(x, q) = f(x, q'))
\]

\[
\iff x \in U(f; t, q').
\]

Hence, \( U(f; t, q) = U(f; t, q') \) for all \( q, q' \in Q \). By Lemma 2.7(3), we have \( U(f; t) = \bigcap_{q \in Q} U(f; t, q) \). By the claim, we have \( U(f; t) = U(f; t, q) \) for all \( q \in Q \). Since \( U(f; t, q) = U(f; t) = I \) for all \( t > k \) and \( U(f; t, q) = U(f; t) = A \) for all \( t \leq k \), it follows from Theorem 2.9(3) that \( f \) is a \( Q \)-fuzzy UP-ideal of \( A \).

\[ \square \]

**Corollary 2.14.** Let \( S \) be a UP-subalgebra of \( A \). Then the following statements hold:

1. For any \( k \in (0, 1] \), there exists a \( Q \)-fuzzy UP-subalgebra \( g \) of \( A \) such that \( L(\g; t) = S \) for all \( t < k \) and \( L(\g; t) = A \) for all \( t \geq k \), and

2. For any \( k \in [0, 1) \), there exists a \( Q \)-fuzzy UP-subalgebra \( f \) of \( A \) such that \( U(f; t) = S \) for all \( t > k \) and \( U(f; t) = A \) for all \( t \leq k \).

**Proof.** (1) Let \( f \) be a \( Q \)-fuzzy set in \( A \) defined by

\[
f(x, q) = \begin{cases} 
  0 & \text{if } x \in S, \\
  k & \text{if } x \notin S,
\end{cases}
\]

for all \( q \in Q \).

In the proof of Corollary 2.13(1) we have \( L(f; t) = S \) for all \( t < k \) and \( L(f; t) = A \) for all \( t \geq k \), and \( L(f; t, q) = L(f; t, q') \) for all \( q, q' \in Q \). By Lemma 2.7(1) we have \( L(f; t) = \bigcap_{q \in Q} L(f; t, q) \). By the claim, we have \( L(f; t) = L(f; t, q) \) for all \( q \in Q \). Since \( L(f; t, q) = L(f; t) = S \) for all \( t < k \) and \( L(f; t, q) = L(f; t) = A \) for all \( t \geq k \), it follows from Theorem 2.11(1) that \( f \) is a \( Q \)-fuzzy UP-subalgebra of \( A \). By Remark 1.17 we have \( L(\g; t) = L(f; t) = S \) for all \( t < k \) and \( L(\g; t) = L(f; t) = A \) for all \( t \geq k \). Let \( \g = g \). Then \( g \) is a \( Q \)-fuzzy UP-subalgebra of \( A \) such that \( L(\g; t) = S \) for all \( t < k \) and \( L(\g; t) = A \) for all \( t \geq k \).

(2) Let \( f \) be a \( Q \)-fuzzy set in \( A \) defined by

\[
f(x, q) = \begin{cases} 
  1 & \text{if } x \in S, \\
  k & \text{if } x \notin S,
\end{cases}
\]

for all \( q \in Q \).
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In the proof of Corollary 2.13(2), we have $U(f; t) = S$ for all $t > k$ and $U(f; t) = A$ for all $t \leq k$, and $U(f; t, q) = U(f; t, q')$ for all $q, q' \in Q$. By Lemma 2.7(3) we have $U(f; t) = \bigcap_{q \in Q} U(f; t, q)$. By the claim, we have $U(f; t) = U(f; t, q)$ for all $q \in Q$. Since $U(f; t, q) = U(f; t) = S$ for all $t > k$ and $U(f; t, q) = U(f; t) = A$ for all $t \leq k$, it follows from Theorem 2.11(3) that $f$ is a Q-fuzzy UP-subalgebra of $A$.

Theorem 2.15. Let $f$ be a Q-fuzzy set in $A$ and $s < t$ for $s, t \in [0, 1]$. Then the following statements hold:

1. $L(f; s, q) = L(f; t, q)$ if and only if there is no $x \in A$ such that $s < f(x, q) \leq t$,
2. $L^{-}(f; s, q) = L^{-}(f; t, q)$ if and only if there is no $x \in A$ such that $s \leq f(x, q) < t$,
3. $U(f; s, q) = U(f; t, q)$ if and only if there is no $x \in A$ such that $s \leq f(x, q) < t$, and
4. $U^{+}(f; s, q) = U^{+}(f; t, q)$ if and only if there is no $x \in A$ such that $s < f(x, q) \leq t$.

Proof. (1) Assume that $L(f; s, q) = L(f; t, q)$. Suppose that there is $x \in A$ such that $s < f(x, q) \leq t$. Then $x \in L(f; t, q)$ but $x \notin L(f; s, q)$, so $L(f; t, q) \neq L(f; s, q)$ which is a contradiction. Hence, there is no $x \in A$ such that $s < f(x, q) \leq t$.

Conversely, assume that there is no $x \in A$ such that $s < f(x, q) \leq t$. Let $x \in L(f; s, q)$. Then $f(x, q) \leq s < t$, so $x \in L(f; t, q)$. Thus $L(f; s, q) \subseteq L(f; t, q)$. Suppose that $L(f; t, q) \not\subseteq L(f; s, q)$. Then there exists $x \in L(f; t, q)$ but $x \notin L(f; s, q)$. Thus $f(x, q) \leq t$ and $f(x, q) > s$, so $s < f(x, q) \leq t$ which is a contradiction. Thus $L(f; t, q) \subseteq L(f; s, q)$. Hence, $L(f; s, q) = L(f; t, q)$.

(2) Similarly to as in the proof of (1).

(3) Assume that $U(f; s, q) = U(f; t, q)$. Suppose that there is $x \in A$ such that $s \leq f(x, q) < t$. Then $x \in U(f; s, q)$ but $x \notin U(f; t, q)$, so $U(f; s, q) \neq U(f; t, q)$ which is a contradiction. Hence, there is no $x \in A$ such that $s \leq f(x, q) < t$.

Conversely, assume that there is no $x \in A$ such that $s \leq f(x, q) < t$. Let $x \in U(f; t, q)$. Then $f(x, q) \geq t > s$, so $x \in U(f; s, q)$. Thus $U(f; t, q) \subseteq U(f; s, q)$. Suppose that $U(f; s, q) \not\subseteq U(f; t, q)$. Then there exists $x \in U(f; s, q)$ but $x \notin U(f; t, q)$. Thus $f(x, q) \geq s$ and $f(x, q) < t$, so $s \leq f(x, q) < t$ which is a contradiction. Thus $U(f; s, q) \subseteq U(f; t, q)$. Hence, $U(f; s, q) = U(f; t, q)$.

(4) Similarly to as in the proof of (3).

Corollary 2.16. Let $f$ be a Q-fuzzy set in $A$ and $s < t$ for $s, t \in [0, 1]$. Then the following statements hold:

1. $L(f; s, q) = L(f; t, q)$ if and only if $U^{+}(f; s, q) = U^{+}(f; t, q)$, and
2. $U(f; s, q) = U(f; t, q)$ if and only if $L^{-}(f; s, q) = L^{-}(f; t, q)$.

Proof. (1) It follows from Theorem 2.15(1) and Theorem 2.15(4).

(2) It follows from Theorem 2.15(2) and Theorem 2.15(3).
Theorem 2.17. Let \((A; \cdot, 0_A)\) and \((B; *, 0_B)\) be UP-algebras and let \(f: A \rightarrow B\) be a UP-homomorphism. Then the following statements hold:

(1) if \(\mu\) is a \(q\)-fuzzy UP-ideal of \(B\), then \(\mu_f\) is also a \(q\)-fuzzy UP-ideal of \(A\), and

(2) if \(\mu\) is a \(q\)-fuzzy UP-subalgebra of \(B\), then \(\mu_f\) is also a \(q\)-fuzzy UP-subalgebra of \(A\).

Proof. (1) Assume that \(\mu\) is a \(q\)-fuzzy UP-ideal of \(B\). Let \(x \in A\). Then

\[
\mu_f(0_A, q) = \mu(f(0_A), q) = \mu(0_B, q) \geq \mu(f(x), q) = \mu_f(x, q).
\]

Let \(x, y, z \in A\). Then

\[
\mu_f(x \cdot z, q) = \mu(f(x \cdot z), q) = \mu(f(x) \cdot f(z), q) \\
\geq \min\{\mu(f(x) \cdot (f(y) \cdot f(z)), q), \mu(f(y), q)\} \geq \min\{\mu(f(x) \cdot (y \cdot z), q), \mu(f(y), q)\} = \min\{\mu_f(x \cdot (y \cdot z), q), \mu_f(y, q)\}.
\]

Hence, \(\mu_f\) is a \(q\)-fuzzy UP-ideal of \(A\).

(2) Assume that \(\mu\) is a \(q\)-fuzzy UP-subalgebra of \(B\). Let \(x, y \in A\). Then

\[
\mu_f(x \cdot y, q) = \mu(f(x \cdot y), q) = \mu(f(x) \cdot f(y), q) \\
\geq \min\{\mu(f(x), q), \mu(f(y), q)\} = \min\{\mu_f(x, q), \mu_f(y, q)\}.
\]

Hence, \(\mu_f\) is a \(q\)-fuzzy UP-subalgebra of \(A\).

With Definition 1.10 and 1.13 and Theorem 2.17, we obtain the corollary.

Corollary 2.18. Let \(f: A \rightarrow B\) be a UP-homomorphism. Then the following statements hold:

(1) if \(\mu\) is a \(Q\)-fuzzy UP-ideal of \(B\), then \(\mu_f\) is also a \(Q\)-fuzzy UP-ideal of \(A\), and

(2) if \(\mu\) is a \(Q\)-fuzzy UP-subalgebra of \(B\), then \(\mu_f\) is also a \(Q\)-fuzzy UP-subalgebra of \(A\).

Theorem 2.19. Let \((A; \cdot, 0_A)\) and \((B; *, 0_B)\) be UP-algebras and let \(f: A \rightarrow B\) be a UP-isomorphism. Then the following statements hold:

(1) if \(\mu_f\) is a \(q\)-fuzzy UP-ideal of \(A\), then \(\mu\) is also a \(q\)-fuzzy UP-ideal of \(B\), and
(2) if $\mu_f$ is a q-fuzzy UP-subalgebra of $A$, then $\mu$ is also a q-fuzzy UP-subalgebra of $B$.

Proof. (1) Assume that $\mu_f$ is a q-fuzzy UP-ideal of $A$. Let $y \in B$. Then there exists $x \in A$ such that $f(x) = y$, we have

$$\mu(0_B, q) = \mu(y * 0_B, q)$$

$$= \mu(f(x) * f(0_A), q)$$

$$= \mu(f(x * 0_A), q)$$

$$= \mu_f(x * 0_A, q)$$

$$\geq \mu_f(x, q)$$

$$= \mu(f(x), q)$$

$$= \mu(y, q).$$

Let $a, b, c \in B$. Then there exist $x, y, z \in A$ such that $f(x) = a$, $f(y) = b$ and $f(z) = c$, we have

$$\mu(a * c, q) = \mu(f(x) * f(z), q)$$

$$= \mu(f(x * z), q)$$

$$= \mu_f(x * z, q)$$

$$\geq \min\{\mu_f(x \cdot (y * z), q), \mu_f(y, q)\}$$

$$= \min\{\mu(f(x \cdot (y * z)), q), \mu_f(y, q)\}$$

$$= \min\{\mu(f(x) \cdot (f(y) \cdot f(z)), q), \mu_f(y, q)\}$$

$$= \min\{\mu(a * (b * c), q), \mu(b, q)\}.$$

Hence, $\mu$ is a q-fuzzy UP-ideal of $B$.

(2) Assume that $\mu_f$ is a q-fuzzy UP-subalgebra of $A$. Let $a, b \in B$. Then there exist $x, y \in A$ such that $f(x) = a$ and $f(y) = b$, we have

$$\mu(a * b, q) = \mu(f(x) \cdot f(y), q)$$

$$= \mu(f(x, y), q)$$

$$= \mu_f(x, y, q)$$

$$\geq \min\{\mu_f(x, q), \mu_f(y, q)\}$$

$$= \min\{\mu(f(x), q), \mu(f(y), q)\}$$

$$= \min\{\mu(a, q), \mu(b, q)\}.$$

Hence, $\mu$ is a q-fuzzy UP-subalgebra of $B$.

With Definition 1.10 and 1.13 and Theorem 2.19 we obtain the corollary.

**Corollary 2.20.** Let $f : A \to B$ be a UP-isomorphism. Then the following statements hold:
(1) if \( \mu_f \) is a \( Q \)-fuzzy UP-ideal of \( A \), then \( \mu \) is also a \( Q \)-fuzzy UP-ideal of \( B \), and

(2) if \( \mu_f \) is a \( Q \)-fuzzy UP-subalgebra of \( A \), then \( \mu \) is also a \( Q \)-fuzzy UP-subalgebra of \( B \).

Lemma 2.21. (Bak\textsuperscript{a}, 2005) For any \( a, b, c, d \in \mathbb{R} \), the following properties hold:

(1) \( \max\{\max\{a, b\}, \max\{c, d\}\} = \max\{\max\{a, c\}, \max\{b, d\}\} \), and

(2) \( \min\{\min\{a, b\}, \min\{c, d\}\} = \min\{\min\{a, c\}, \min\{b, d\}\} \).

Let \((A; \cdot, 0_A)\) and \((B; \ast, 0_B)\) be UP-algebras. We can easily prove that \( A \times B \) is a UP-algebra defined by

\[
(x_1, x_2) \circ (y_1, y_2) = (x_1 \cdot y_1, x_2 \ast y_2)
\]

for all \( x_1, y_1 \in A \) and \( x_2, y_2 \in B \).

Theorem 2.22. Let \((A; \cdot, 0_A)\) and \((B; \ast, 0_B)\) be UP-algebras. Then the following statements hold:

(1) if \( \mu \) is a \( Q \)-fuzzy UP-ideal of \( A \) and \( \delta \) is a \( q \)-fuzzy UP-ideal of \( B \), then \( \mu \cdot \delta \) is a \( Q \)-fuzzy UP-ideal of \( A \times B \), and

(2) if \( \mu \) is a \( Q \)-fuzzy UP-subalgebra of \( A \) and \( \delta \) is a \( q \)-fuzzy UP-subalgebra of \( B \), then \( \mu \cdot \delta \) is a \( Q \)-fuzzy UP-subalgebra of \( A \times B \).

Proof. (1) Assume that \( \mu \) is a \( Q \)-fuzzy UP-ideal of \( A \) and \( \delta \) is a \( q \)-fuzzy UP-ideal of \( B \). Let \((x_1, x_2) \in A \times B \). Then

\[
(\mu \cdot \delta)((0_A, 0_B), q) = \min\{\mu(0_A, q), \delta(0_B, q)\} \geq \min\{\mu(x_1, q), \delta(x_2, q)\} = (\mu \cdot \delta)((x_1, x_2), q).
\]

Let \((x_1, x_2), (y_1, y_2), (z_1, z_2) \in A \times B \). Then

\[
(\mu \cdot \delta)((x_1, x_2) \circ (z_1, z_2), q)
\]

\[
= (\mu \cdot \delta)((x_1 \cdot z_1, x_2 \ast z_2), q)
\]

\[
= \min\{\mu(x_1, z_1), \delta(x_2, z_2)\} \geq \min\{\min\{\mu(x_1, (y_1 \cdot z_1)), \mu(y_1, q)\}, \min\{\mu((x_1 \cdot (y_1 \cdot z_1)), (y_1 \cdot z_1, y_2 \ast z_2), q), \delta(y_2, q)\}\}
\]

\[
= \min\{\min\{\mu(x_1, (y_1 \cdot z_1)), \delta(x_2, (y_2 \ast z_2)), \mu(y_1, q)\}, \min\{\mu(y_1, (y_1 \cdot z_1)), \delta(y_2, (y_2 \ast z_2)), \mu(y_1, q)\}\}
\]

\[
= \min\{\min\{\mu \cdot \delta)((x_1, x_2) \circ (y_1 \cdot z_1, y_2 \ast z_2)), q\}, (\mu \cdot \delta)((y_1, y_2), q)\}
\]

Hence, \( \mu \cdot \delta \) is a \( Q \)-fuzzy UP-ideal of \( A \times B \).
(2) Assume that $\mu$ is a $q$-fuzzy UP-subalgebra of $A$ and $\delta$ is a $q$-fuzzy UP-subalgebra of $B$. Let $(x_1, x_2), (y_1, y_2) \in A \times B$. Then

$$(\mu \cdot \delta)((x_1, x_2) \diamond (y_1, y_2), q)$$

$$= (\mu \cdot \delta)((x_1 \cdot y_1, x_2 \ast y_2), q)$$

$$= \min\{\mu(x_1 \cdot y_1, q), \delta(x_2 \ast y_2, q)\}$$

$$\geq \min\{\min\{\mu(x_1, q), \mu(y_1, q)\}, \min\{\delta(x_2, q), \delta(y_2, q)\}\}$$

$$= \min\{\min\{\mu(x_1, q), \delta(x_2, q)\}, \min\{\mu(y_1, q), \delta(y_2, q)\}\}$$

$$= \min\{\mu \cdot \delta\}(x_1, x_2, q), (\mu \cdot \delta)(y_1, y_2, q)\}.$$  

Hence, $\mu \cdot \delta$ is a $q$-fuzzy UP-subalgebra of $A \times B$. \qed

Give examples of conflict that $\mu$ and $\delta$ are $q$-fuzzy UP-ideals (resp. $q$-fuzzy UP-subalgebras) of $A$ but $\mu \times \delta$ is not a $q$-fuzzy UP-ideal (resp. $q$-fuzzy UP-subalgebra) of $A \times A$.

**Example 2.23.** Let $A = \{0, 1\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Then $(A; \cdot, 0)$ is a UP-algebra. Let $Q = \{q\}$. We define $Q$-fuzzy sets $\mu$ and $\delta$ in $A$ as follows: $\mu(0, q) = 0.2, \delta(0, q) = 0.3, \mu(1, q) = 0.1$ and $\delta(1, q) = 0.1$. Using this data, we can show that $\mu$ and $\delta$ are $q$-fuzzy UP-ideals of $A$. Let $(x_1, x_2) = (0, 0), (y_1, y_2) = (1, 0), (z_1, z_2) = (1, 1) \in A \times A$. Then

$$(\mu \times \delta)((x_1, x_2) \diamond (z_1, z_2), q) = 0.1$$

and

$$\min\{\mu \times \delta\}((x_1, x_2) \diamond [y_1, y_2] \diamond (z_1, z_2), q), (\mu \times \delta)((y_1, y_2), q)\} = 0.2.$$ 

Hence, $(\mu \times \delta)((x_1, x_2) \diamond (z_1, z_2), q) \neq \min\{\mu \times \delta\}((x_1, x_2) \diamond [y_1, y_2] \diamond (z_1, z_2), q), (\mu \times \delta)((y_1, y_2), q)\}$. Therefore, $\mu \times \delta$ is not a $q$-fuzzy UP-ideal of $A \times A$.

**Example 2.24.** Let $A = \{0, 1, 2\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $(A; \cdot, 0)$ is a UP-algebra. Let $Q = \{q\}$. We defined a $Q$-fuzzy set $\mu$ and $\delta$ in $A$ as follows: $\mu(0, q) = 0.4, \delta(0, q) = 0.7, \mu(1, q) = 0.1, \delta(1, q) = 0.1, \mu(2, q) = 0.3$ and $\delta(2, q) = 0.3$. Using this data, we can show that $\mu$ and $\delta$ are $q$-fuzzy UP-subalgebras of $A$. Let $(x_1, x_2) = (0, 1), (y_1, y_2) = (1, 2) \in A \times A$. Then

$$(\mu \times \delta)((x_1, x_2) \diamond (y_1, y_2), q) = 0.1$$
and
\[ \min\{(\mu \times \delta)((x_1, x_2), q), (\mu \times \delta)((y_1, y_2), q)\} = 0.3. \]

Hence, \((\mu \times \delta)((x_1, x_2) \circ (y_1, y_2), q) \ngeq \min\{(\mu \times \delta)((x_1, x_2), q), (\mu \times \delta)((y_1, y_2), q)\} \).

Therefore, \(\mu \times \delta\) is not a \(q\)-fuzzy UP-subalgebra of \(A \times A\).

With Definition 1.10 and 1.13 and Theorem 2.22 we obtain the corollary.

**Corollary 2.25.** The following statements hold:

1. If \(\mu\) is a \(Q\)-fuzzy UP-ideal of \(A\) and \(\delta\) is a \(Q\)-fuzzy UP-ideal of \(B\), then \(\mu \cdot \delta\) is a \(Q\)-fuzzy UP-ideal of \(A \times B\), and

2. If \(\mu\) is a \(Q\)-fuzzy UP-subalgebra of \(A\) and \(\delta\) is a \(Q\)-fuzzy UP-subalgebra of \(B\), then \(\mu \cdot \delta\) is a \(Q\)-fuzzy UP-subalgebra of \(A \times B\).

**Theorem 2.26.** If \(\mu\) is a \(Q\)-fuzzy set in \(A\) and \(\delta\) is a \(Q\)-fuzzy set in \(B\) such that \(\mu \cdot \delta\) is a \(Q\)-fuzzy UP-ideal of \(A \times B\), then the following statements hold:

1. Either \(\mu(0_A, q) \geq \mu(x, q)\) for all \(x \in A\) or \(\delta(0_B, q) \geq \delta(x, q)\) for all \(x \in B\),

2. Either \(\mu(0_A, q) \geq \mu(x, q)\) for all \(x \in A\), then either \(\delta(0_B, q) \geq \mu(x, q)\) for all \(x \in A\) or \(\delta(0_B, q) \geq \delta(x, q)\) for all \(x \in B\), and

3. If \(\delta(0_B, q) \geq \delta(x, q)\) for all \(x \in B\), then either \(\mu(0_A, q) \geq \mu(x, q)\) for all \(x \in A\) or \(\mu(0_A, q) \geq \delta(x, q)\) for all \(x \in B\).

**Proof.** (1) Suppose that there exist \(x \in A\) and \(y \in B\) such that \(\mu(0_A, q) < \mu(x, q)\) and \(\delta(0_B, q) < \delta(y, q)\). Then
\[
(\mu \cdot \delta)((x, y), q) = \min\{\mu(x, q), \delta(y, q)\}
\geq \min\{\mu(0_A, q), \delta(0_B, q)\}
= (\mu \cdot \delta)((0_A, 0_B), q)
\]
which is a contradiction. Hence, \(\mu(0_A, q) \geq \mu(x, q)\) for all \(x \in A\) or \(\delta(0_B, q) \geq \delta(x, q)\) for all \(x \in B\).

(2) Assume that \(\mu(0_A, q) \geq \mu(x, q)\) for all \(x \in A\). Suppose that there exist \(x \in A\) and \(y \in B\) such that \(\delta(0_B, q) < \mu(x, q)\) and \(\delta(0_B, q) < \delta(y, q)\). Then \(\mu(0_A, q) \geq \mu(x, q) > \delta(0_B, q)\). Thus
\[
(\mu \cdot \delta)((x, y), q) = \min\{\mu(x, q), \delta(y, q)\}
\geq \min\{\delta(0_B, q), \delta(0_B, q)\}
= \delta(0_B, q)
= \min\{\mu(0_A, q), \delta(0_B, q)\}
= (\mu \cdot \delta)((0_A, 0_B), q)
\]
which is a contradiction. Hence, \(\delta(0_B, q) \geq \mu(x, q)\) for all \(x \in A\) or \(\delta(0_B, q) \geq \delta(x, q)\) for all \(x \in B\).
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(3) Assume that $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$. Suppose that there exist $x \in A$ and $y \in B$ such that $\mu(0_A, q) < \mu(x, q)$ and $\mu(0_A, q) < \delta(y, q)$. Then $\delta(0_B, q) \geq \delta(x, q) > \mu(0_A, q)$. Thus

$$(\mu \cdot \delta)((x, y), q) = \min\{\mu(x, q), \delta(y, q)\}$$

$$> \min\{\mu(0_A, q), \mu(0_A, q)\}$$

$$= \mu(0_A, q)$$

$$= \min\{\mu(0_A, q), \delta(0_B, q)\}$$

$$= (\mu \cdot \delta)((0_A, 0_B), q)$$

which is a contradiction. Hence, $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$ or $\mu(0_A, q) \geq \delta(x, q)$ for all $x \in B$.

With Definition 1.10 and 1.13 and Theorem 2.26, we obtain the corollary.

**Corollary 2.27.** If $\mu$ is a Q-fuzzy set in $A$ and $\delta$ is a Q-fuzzy set in $B$ such that $\mu \cdot \delta$ is a Q-fuzzy UP-ideal of $A \times B$, then the following statements hold:

1. for all $q \in Q$, either $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$ or $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$,

2. for all $q \in Q$, if $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$, then either $\delta(0_B, q) \geq \mu(x, q)$ for all $x \in A$ or $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$, and

3. for all $q \in Q$, if $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$, then either $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$ or $\mu(0_A, q) \geq \delta(x, q)$ for all $x \in B$.

**Theorem 2.28.** Let $(A; *, 0_A)$ and $(B; *, 0_B)$ be UP-algebras and let $\mu$ be a Q-fuzzy set in $A$ and $\delta$ be a Q-fuzzy set in $B$. Then the following statements hold:

1. if $\mu \cdot \delta$ is a q-fuzzy UP-ideal of $A \times B$, then either $\mu$ is a q-fuzzy UP-ideal of $A$ or $\delta$ is a q-fuzzy UP-ideal of $B$, and

2. if $\mu \cdot \delta$ is a q-fuzzy UP-subalgebra of $A \times B$, then either $\mu$ is a q-fuzzy UP-subalgebra of $A$ or $\delta$ is a q-fuzzy UP-subalgebra of $B$.

**Proof.** (1) Assume that $\mu \cdot \delta$ is a q-fuzzy UP-ideal of $A \times B$. Suppose that $\mu$ is not a q-fuzzy UP-ideal of $A$ and $\delta$ is not a q-fuzzy UP-ideal of $B$. By Theorem 2.26(1), we have $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$ or $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$. Suppose that $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$. By Theorem 2.26(2), either $\delta(0_B, q) \geq \mu(x, q)$ for all $x \in A$ or $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$. If $\delta(0_B, q) \geq \mu(x, q)$ for all $x \in A$, then
\[(\mu \cdot \delta)((x, 0_B), q) = \min\{\mu(x, q), \delta(0_B, q)\} = \mu(x, q).\] We consider, for all \(x, y, z \in A,

\[
\mu(x \cdot z, q) = \min\{\mu(x \cdot z, q), \delta(0_B, q)\}
\[
= (\mu \cdot \delta)((x \cdot z, 0_B), q)
\[
= (\mu \cdot \delta)((x \cdot z, 0_B \ast 0_B), q)
\[
= (\mu \cdot \delta)((x, 0_B) \circ (z, 0_B), q)
\[
\geq \min\{(\mu \cdot \delta)((x, 0_B) \circ (y, 0_B) \circ (z, 0_B)), q)\],
\[
(\mu \cdot \delta)((y, 0_B), q)
\[
= \min\{(\mu \cdot \delta)((x \cdot y, 0_B \ast 0_B), q), (\mu \cdot \delta)((y, 0_B), q)\}
\[
= \min\{\min\{\mu(x \cdot (y \cdot z), q), \delta(0_B, q)\},
\[
\min\{\mu(y, q), \delta(0_B, q)\}\}
\[
= \min\{\mu(x \cdot (y \cdot z), q), \mu(y, q)\}.
\]

Hence, \(\mu\) is a \(q\)-fuzzy UP-ideal of \(A\) which is a contradiction. Suppose that \(\delta(0_B, q) \geq \delta(x, q)\) for all \(x \in B\). By Theorem 2.26(3), either \(\mu(0_A, q) \geq \mu(x, q)\) for all \(x \in A\) or \(\mu(0_A, q) \geq \delta(x, q)\) for all \(x \in B\). If \(\mu(0_A, q) \geq \delta(x, q)\) for all \(x \in B\), then

\[
\delta(x \ast z, q) = \min\{\mu(0_A, q), \delta(x \ast z, q)\}
\[
= (\mu \cdot \delta)((0_A, x \ast z), q)
\[
= (\mu \cdot \delta)((0_A, 0_A, x \ast z), q)
\[
= (\mu \cdot \delta)((0_A, x) \circ (0_A, z), q)
\[
\geq \min\{\mu \cdot \delta)((0_A, x) \circ (0_A, y) \circ (0_A, z), q),
\[
(\mu \cdot \delta)((0_A, y), q)
\[
= \min\{\mu \cdot \delta)((0_A \cdot (0_A \cdot 0_A), x \ast (y \ast z)), q), (\mu \cdot \delta)((0_A, y), q)\}
\[
= \min\{\mu \cdot \delta)((0_A, x \ast (y \ast z)), q),
\[
(\mu \cdot \delta)((0_A, y), q)
\[
= \min\{\min\{\mu(0_A, q), \delta(x \ast (y \ast z), q)\},
\[
\min\{\mu(0_A, q), \delta(y, q)\}\}
\[
= \min\{\delta(x \ast (y \ast z), q), \delta(y, q)\}.
\]

Hence, \(\delta\) is a \(q\)-fuzzy UP-ideal of \(B\) which is a contradiction. Since \(\mu\) is not a \(q\)-fuzzy UP-ideal of \(A\) and \(\delta\) is not a \(q\)-fuzzy UP-ideal of \(B\), we have \(\mu(0_A, q) \geq \mu(x, q)\) for all \(x \in A\) and \(\delta(0_B, q) \geq \delta(x, q)\) for all \(x \in B\). Let \(x_1, x_2, x_3 \in A\) and \(y_1, y_2, y_3 \in B\) be such that \(\mu(x_1 \cdot x_3, q) < \min\{\mu(x_1 \cdot (x_2 \cdot x_3), q), \mu(x_2, q)\}\) and \(\delta(y_1 \ast y_3, q) < \min\{\delta(y_1 \ast (y_2 \ast y_3), q), \delta(y_2, q)\}\), so min\{\(\mu(x_1 \cdot x_3, q), \delta(y_1 \ast y_3, q)\} <
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\[
\min\{\min\{\mu(x_1 \cdot (x_2 \cdot x_3), q), \mu(x_2, q)\}, \min\{\delta(y_1 \cdot (y_2 \cdot y_3), q), \delta(y_2, q)\}\}. \quad \text{Thus}
\]

\[
\min\{\min\{\mu(x_1 \cdot x_3, q), \delta(y_1 \cdot y_3, q)\}
\]
\[
= (\mu \cdot \delta)((x_1 \cdot x_3, y_1 \cdot y_3), q)
\]
\[
= (\mu \cdot \delta)((x_1, y_1) \circ (x_3, y_3), q)
\]
\[
\geq \min\{(\mu \cdot \delta)((x_1, y_1) \circ [(x_2, y_2) \circ (x_3, y_3)], q),
\]
\[
(\mu \cdot \delta)((x_2, y_2), q)\} \quad \text{(Definition 1.25)}
\]
\[
= \min\{(\mu \cdot \delta)((x_1 \cdot (x_2 \cdot x_3), y_1 \cdot (y_2 \cdot y_3)), q), (\mu \cdot \delta)((x_2, y_2), q)\}
\]
\[
= \min\{\min\{\mu(x_1 \cdot (x_2 \cdot x_3), q), \delta(y_1 \cdot (y_2 \cdot y_3), q)\},
\]
\[
\min\{\mu(x_2, q), \delta(y_2, q)\}\} \quad \text{(Definition 1.25)}
\]
\[
= \min\{\min\{\mu(x_1 \cdot (x_2 \cdot x_3), q), \mu(x_2, q)\},
\]
\[
\min\{\delta(y_1 \cdot (y_2 \cdot y_3), q), \delta(y_2, q)\}\}. \quad \text{(Lemma 2.21)}
\]

It follows that \(\min\{\mu(x_1 \cdot x_3, q), \delta(y_1 \cdot y_3, q)\} \neq \min\{\min\{\mu(x_1 \cdot (x_2 \cdot x_3), q), \mu(x_2, q)\}, \min\{\delta(y_1 \cdot (y_2 \cdot y_3), q), \delta(y_2, q)\}\} \) which is a contradiction. Hence, \(\mu\) is a \(q\)-fuzzy UP-ideal of \(A\) or \(\delta\) is a \(q\)-fuzzy UP-ideal of \(B\).

(2) Assume that \(\mu \cdot \delta\) is a \(q\)-fuzzy UP-subalgebra of \(A \times B\). Suppose that \(\mu\) is not a \(q\)-fuzzy UP-subalgebra of \(A\) and \(\delta\) is not a \(q\)-fuzzy UP-subalgebra of \(B\). Then there exist \(x, y \in A\) and \(a, b \in B\) such that

\[
\mu(x \cdot y, q) < \min\{\mu(x, q), \mu(y, q)\} \quad \text{and} \quad \delta(a \cdot b, q) < \min\{\delta(a, q), \delta(b, q)\}.
\]

Then \(\min\{\mu(x \cdot y, q), \delta(a \cdot b, q)\} < \min\{\min\{\mu(x, q), \mu(y, q)\}, \min\{\delta(a, q), \delta(b, q)\}\}. \quad \text{Consider,}
\]

\[
\min\{\mu(x \cdot y, q), \delta(a \cdot b, q)\} = (\mu \cdot \delta)((x \cdot y, a \cdot b), q)
\]
\[
= (\mu \cdot \delta)((x, a) \circ (y, b), q)
\]
\[
\geq \min\{(\mu \cdot \delta)((x, a), q),
\]
\[
(\mu \cdot \delta)((y, b), q)\} \quad \text{(Definition 1.13)}
\]
\[
= \min\{\min\{\mu(x, q), \delta(a, q)\},
\]
\[
\min\{\mu(y, q), \delta(b, q)\}\} \quad \text{(Definition 1.25)}
\]
\[
= \min\{\min\{\mu(x, q), \mu(y, q)\},
\]
\[
\min\{\delta(a, q), \delta(b, q)\}\}. \quad \text{(Lemma 2.21)}
\]

Thus \(\min\{\mu(x \cdot y, q), \delta(a \cdot b, q)\} \neq \min\{\min\{\mu(x, q), \mu(y, q)\}, \min\{\delta(a, q), \delta(b, q)\}\} \) which is a contradiction. Hence, \(\mu\) is a \(q\)-fuzzy UP-subalgebra of \(A\) or \(\delta\) is a \(q\)-fuzzy UP-subalgebra of \(B\).

Give examples of conflict that \(\mu\) and \(\delta\) are not \(Q\)-fuzzy UP-ideals (resp. \(Q\)-fuzzy UP-subalgebras) of \(A\) but \(\mu \cdot \delta\) is a \(Q\)-fuzzy UP-ideal (resp. \(Q\)-fuzzy UP-subalgebra) of \(A \times A\).
Example 2.29. Let $A = \{0, 1\}$ be a set with a binary operation $\cdot$ defined by the following table:

\[
\begin{array}{c|cc}
. & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{array}
\]

Then $(A; \cdot, 0)$ is a UP-algebra. Let $Q = \{a, b\}$. We define two $Q$-fuzzy sets $\mu$ and $\delta$ in $A$ as follows:

\[
\begin{array}{c|cc}
\mu & a & b \\
\hline
0 & 0.1 & 0.3 \\
1 & 0.3 & 0.3 \\
\end{array}
\]

and

\[
\begin{array}{c|cc}
\delta & a & b \\
\hline
0 & 0.3 & 0.1 \\
1 & 0.3 & 0.3 \\
\end{array}
\]

Since $\mu(0, a) = 0.1 < 0.3 = \mu(1, a)$, we have $\mu(0, a) \not\geq \mu(1, a)$. Thus $\mu$ is not an $a$-fuzzy UP-ideal of $A$. Since $\delta(0, b) = 0.1 < 0.3 = \delta(1, b)$, we have $\delta(0, b) \not\geq \delta(1, b)$. Thus $\delta$ is not a $b$-fuzzy UP-ideal of $A$. Therefore, $\mu$ and $\delta$ are not $Q$-fuzzy UP-ideals of $A$. Using the above data, we can show that $\mu \cdot \delta$ is a $Q$-fuzzy UP-ideal of $A \times A$.

Example 2.30. Let $A = \{0, 1\}$ be a set with a binary operation $\cdot$ defined by the following table:

\[
\begin{array}{c|cc}
. & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{array}
\]

Then $(A; \cdot, 0)$ is a UP-algebra. Let $Q = \{a, b\}$. We define two $Q$-fuzzy sets $\mu$ and $\delta$ in $A$ as follows:

\[
\begin{array}{c|cc}
\mu & a & b \\
\hline
0 & 0.1 & 0.3 \\
1 & 0.3 & 0.3 \\
\end{array}
\]

and

\[
\begin{array}{c|cc}
\delta & a & b \\
\hline
0 & 0.3 & 0.1 \\
1 & 0.3 & 0.3 \\
\end{array}
\]

Since $\mu(1 \cdot 1, a) = \mu(0, a) = 0.1 < 0.3 = \min\{0.3, 0.3\} = \min\{\mu(1, a), \mu(1, a)\}$, we have $\mu(1 \cdot 1, a) \not\geq \min\{\mu(1, a), \mu(1, a)\}$. Thus $\mu$ is not an $a$-fuzzy UP-subalgebra of $A$. Since $\delta(1 \cdot 1, b) = \delta(0, b) = 0.1 < 0.3 = \min\{0.3, 0.3\} = \min\{\delta(1, b), \delta(1, b)\}$, we have $\delta(1 \cdot 1, b) \not\geq \min\{\delta(1, b), \delta(1, b)\}$. Thus $\delta$ is not a $b$-fuzzy UP-subalgebra of $A$. Therefore, $\mu$ and $\delta$ are not $Q$-fuzzy UP-subalgebras of $A$. By Example 2.29, we have $\mu \cdot \delta$ is a $Q$-fuzzy UP-ideal of $A \times A$. By Corollary 2.2, we have $\mu \cdot \delta$ is a $Q$-fuzzy UP-subalgebra of $A \times A$.

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References


