## Analytical Explicit Formulas of Average Run Length for Long Memory Process with ARFIMA Model on CUSUM Control Chart

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Original Article

Analytical Explicit Formulas of Average Run Length for Long Memory Process with ARFIMA Model on CUSUM Control Chart

Wilasinee Peerajit\textsuperscript{1}, Yupaporn Areepong\textsuperscript{1*}, and Saowanit Sukparungsee\textsuperscript{1}

\textsuperscript{1}Department of Applied Statistics, Faculty of Applied Science, King Mongkut’s University of Technology North Bangkok, Bangkok, 10800, Thailand

* Corresponding author, Email address: yupaporn.a@sci.kmutnb.ac.th

Abstract

This paper proposes the explicit formulas for the derivation of exact formulas from Average Run Lengths (ARLs) using integral equation on CUSUM control chart when observations are long memory processes with exponential white noise. The authors compared efficiency in terms of the percentage of absolute difference to a similar method to verify the accuracy of the ARLs between the values obtained by the explicit formulas and numerical integral equation (NIE) method. The explicit formulas were based on Banach fixed point theorem which was used to guarantee the existence and uniqueness of the solution for ARFIMA($p,d,q$). Results showed that the two methods are similar in good agreement with the percentage of absolute difference at less than 0.23\%. Therefore, the explicit formulas are an efficient alternative for implementation in real applications because the computational CPU time for ARLs from the explicit formulas are 1 second preferable over the NIE method.

Keywords: ARFIMA($p,d,q$) process, Numerical Integral Equation (NIE) method, Exponential white noise.
1. Introduction

CUSUM control chart was first introduced by Page (1954) and has been continually developed by many researchers e.g. (Ewan and Kemp, 1960; Johnson and Leone, 1962; Ewan, 1963; Bissell, 1969; Lucas, 1976; Ryan, 1989; Hawkins and Olwell, 1998). These are commonly used instead of the Shewhart chart as they directly incorporate all of the information in the sequence from the values and detect small shifts in the mean more quickly and can widely implement control processes. Usually, this involves an evaluation of the control chart performance based on the Average Run Lengths (ARLs).

Average Run Length (ARL) is the expected number of observation taken from an in-control process until the control chart falsely signals out-of-control. ARL, as a common characteristic, is widely used as a measure of performance of a control chart. Ideally, the ARL is large enough to keep the level of false alarms at an acceptable level. \( \text{ARL}_0 \) is the notation for the in-control Average Run Length. The out-of-control Average Run Length is denoted by Average Delay for the out-of-control process (\( \text{ARL}_1 \)). It is defined as the expectation of delay time for a true alarm. This time should minimize the quantity as possible.

The used ARLs have been widely applied to techniques of control charts, percentage points have also been recommended (see, for example (Barnard, 1954; Bissell, 1969)). Evaluate ARLs for the CUSUM control charts (including, for example (Page, 1954; Ewan and Kemp, 1960; Goel and Wu, 1971; Brook and Evans, 1972; Woodall, 1983; Fellner, 1990; Hawkins, 1992; Gan, 1992; Luceño and Puig-Pey, 2000; Luceño and Puig-Pey, 2002)) have been conducted.
The integral equation is encountered in a variety of applications from many fields including continuum mechanics, mathematical economics, queuing theory, potential theory, geophysics, electricity and magnetism, optimization, optimal control systems, communication theory, population genetics, medicine etc. The integral equation was provided by Page (1954) and was used to approximate the ARLs of control chart by assuming as a small shift in mean. A computation program based on the integral equation procedure was given by Vance (1986). Goel and Wu (1971) provided a nomogram for the determination of chart parameters of a CUSUM control chart. Lashkari and Rahim (1982) and Chung (1992) found economic designs of CUSUM control charts.

The model of autoregressive fractionally integrated moving average (ARFIMA) processes have a fractional differencing parameter \( d \) which are used to model a long memory (or is so-called long range) and stationary and invertible when values of \( d \) take between (-0.5, 0.5). These processes were introduced by Granger and Joyeux (1980) and Hosking (1981), a detailed description of long memory processes can be found in; e.g. (Baran, 1994; Baillie, 1996; Palma, 2007; Proietti, 2014). The long memory process is involved in a number of applications including finance and economics, environmental, science and engineering. Control charts have been used to combine the long memory process with time series. The control chart is necessary as it is a number of time series following the ARFIMA model, papers by Ramjee (2000) also analyzed the performance of Shewhart and EWMA control charts for the presence of correlated data which occurred from an ARFIMA model. The study result showed that these charts cannot perform well when detecting process shifts. And thus, a new type of control chart and Hyperbolic Weighted Moving Average (HWMA) control chart was proposed. Two
years later, Ramjee et al. (2002) introduced a HWMA control chart forecast-based control chart, specially designed for non-stationary ARFIMA models. Caballero et al. (2002) performed a number of tests on the analysis of daily time series of mid-latitude near-surface air temperature by plotting long-range dependent processes. Furthermore, Pan and Chen (2008) studied control chart for autocorrelated data using ARFIMA model to monitor the long memory air quality data for comparison. As a result, the residual control charts by using ARFIMA models were more appropriate than ARIMA models. Recently, Rabyk and Schmid (2016) introduced EWMA control charts to detect changes in the mean of a long-memory process.

Exponential white noise coordinated with time series has also been investigated. Jacob and Lewis (1977) analyzed autoregressive moving average process order (1,1) denoted by ARMA(1,1) when observations are Exponentially distributed with exponential white noise. The exponential white noise was also used Bayesian methods to analyze the autoregressive model as proposed by Mohamed and Hocine (2010).

Several techniques to evaluate ARLs for the CUSUM and EWMA control charts including Monte Carlo simulations (MC), Markov Chain approach (MCA), numerical integral equation (NIE) method and explicit formulas have been proposed in the previous literature. For example, The Markov Chain approach (MCA) was introduced by Brook and Evans (1972), and many researchers have been studied this matter, in particular, Champ and Woodall (1987), and Champ and Rigdon (1991). Gan (1992) and Gan (1996) presented an accurate NIE method based on an integral equation to compute the ARLs of CUSUM charts under linear trends. Recently, Areepong (2009) proposed analytical derivation to find explicit formulas of ARLs for EWMA control
chart when observations are Exponential distribution. For example, problems from mathematical explicit formulas of ARLs using Fredholm integral equation for one-sided EWMA control chart with Laplace distribution and CUSUM control chart with Hyperexponential distribution was presented by Miti telu et al. (2010). Busaba et al. (2012) analyzed the explicit formulas of ARLs for CUSUM control chart in cases of stationary first order autoregressive: AR(1) process with exponential white noise. The numerical integral equation (NIE) method of ARLs and solution to the numerically using the Gauss-Legendre numerical integral equations was derived by Petcharat et al. (2012) when observations are the first order of moving average process, MA(1), with exponential white noise. Phanyaem et al. (2014) studied analytical exact formulas of ARL$_0$ and ARL$_1$ using integral equation and NIE method for CUSUM control chart for ARMA(1,1) process with exponential distribution white noise. Recently, Petcharat et al. (2015) derived the explicit formulas of ARLs for CUSUM control chart when observations are the $q$ order moving average, MA($q$) with exponential white noise by using the integral equation. The integral equation was based on Fredholm integral equations of the second kind. Finally, Peerajit et al. (2016) presented the numerical integral equation (NIE) method of ARLs on CUSUM control chart for long memory process with an ARFIMA model with exponential white noise.

The aim of this paper is to present the explicit formulas and numerical integral equation (NIE) method for ARFIMA process. In section 2, the long memory process for ARFIMA model on CUSUM control chart is presented. In section 3, the uniqueness of solution by using Banach fixed point theorem is described (Venkateshwara et al., 2001). In section 4, the solutions of the integral equation for ARLs are presented and the comparison of analytical results between explicit formulas and NIE method are
presented in section 5. Finally, section 6 summarized the real applications in this paper along with a few topics for further research.

2. The Long Memory Process for ARFIMA Model on CUSUM Control Chart

The CUSUM control chart was the first introduced by Page (1954) to detect small shifts in the mean of a process and is now widely implemented in process control. Let $\xi_t$ be observations of a stationary autoregressive fractionally integrated moving average (ARFIMA) process of order $(p,d,q)$, denoted by ARFIMA$(p,d,q)$ with exponential white noise. The ARFIMA$(p,d,q)$ process shows the characteristic of long memory when the parameter $d$ (the degree of differencing) takes values between $(0, 0.5)$, see. (Granger and Joyeux, 1980; Hosking, 1981; Baillie, 1996).

The general form of the ARFIMA$(p,d,q)$ process $(X_t)$ which is used on CUSUM control charts has the following form:

$$
X_t = \mu + \xi_t - \theta_1 \xi_{t-1} - \theta_2 \xi_{t-2} - \ldots - \theta_q \xi_{t-q}
- \left( -dX_{t-1} + \frac{d(d-1)}{2!}X_{t-2} - \frac{d(d-1)(d-2)}{3!}X_{t-3} + \ldots \right)
+ \left( \phi_1 X_{t-1} - d\phi_1 X_{t-2} + \frac{d(d-1)}{2!}\phi_1 X_{t-3} - \frac{d(d-1)(d-2)}{3!}\phi_1 X_{t-4} + \ldots \right)
+ \left( \phi_2 X_{t-2} - d\phi_2 X_{t-3} + \frac{d(d-1)}{2!}\phi_2 X_{t-4} - \frac{d(d-1)(d-2)}{3!}\phi_2 X_{t-5} + \ldots \right)
\vdots
+ \left( \phi_p X_{t-p} - d\phi_p X_{t-p-1} + \frac{d(d-1)}{2!}\phi_p X_{t-p-2} - \frac{d(d-1)(d-2)}{3!}\phi_p X_{t-p-3} + \ldots \right),
$$

where $\xi_t$ is a white noise process assumed with exponential distribution ($\xi_t \sim \text{Exp}(\alpha)$).

The initial value is normally the process mean, $|\phi_i|<1$ is an autoregressive coefficient; $i=1,2,\ldots, p$ and $|\theta_i|<1$ is a moving average coefficient; $i=1,2,\ldots, q$. It is
assumed the initial value of (ARFIMA\((p,d,q)\) processes \(\xi_{t-1}, \xi_{t-2}, \ldots, \xi_{t-q}, X_{t-1}, \ldots\)) \(X_{t-p}, X_{t-p-1}, \ldots\) equal 1 and \(\mu\) is a constant.

The CUSUM chart based on ARFIMA\((p,d,q)\) process is defined by the following recursion:

\[ Y_t = \max(Y_{t-1} + X_t - a, 0), \quad t = 1, 2, \ldots, Y_0 = u, \quad (2) \]

where \(Y_t\) is the CUSUM statistic, \(X_t\) is a sequence of ARFIMA\((p,d,q)\) process, the starting value \(Y_0 = u\) is an initial value and \(a\) is a reference value of CUSUM chart.

The corresponding stopping time \((\tau_b)\) for (2) is defined as:

\[ \tau_b = \inf \{ t > 0; \ Y_t > b \}, \quad u < b, \quad (3) \]

where \(b\) is a constant on known parameter as the Upper Control Limit (UCL).

3. Uniqueness of Solution of Integral Equation for the ARLs

The ARLs of the CUSUM control chart are defined as \(C(u) = E_u(\tau_b)\). The notation \(P_y\) denote the probability measure, the notation \(E_y\) denote the induced expectation corresponding to the initial value \(Y_0 = u\), and \(C(u)\) denote the ARLs of ARFIMA process on CUSUM chart. Then the function of \(C(u)\) is initial value \(u; u \in [0, b]\), which can be shown by the ARLs (Venkateshwara et al., 2001; Mititelu et al., 2010), defined as \(\text{ARL} = C(u) = E_u(\tau_b) < \infty\), is the unique of solution of integral equation for ARLs as follows:

\[ C(u) = 1 + E_y[I\{0 < Y_1 < b\}C(Y_1)] + P_y\{Y_1 = 0\}C(0), \quad (4) \]

where
\( I(0 < Y_i < b) = \begin{cases} 
1 & ; 0 < Y_i < b \\
0 & ; Otherwise 
\end{cases} \)

is the indicator function.

Let \( \xi_t \) is continuous distribution i.i.d random variable with exponential distributed given by \( F(u) = 1 - e^{-\alpha u} \) and \( f(u) = \frac{dF(u)}{du} = \alpha e^{-\alpha u} \) have been proposed in (Mititelu et al., 2010; Mititelu et al., 2011). Hence, the integral equation of ARFIMA\((p,d,q)\) process on CUSUM control chart can be written in the below form:

\[
C(u) = 1 + \alpha e^{\alpha (a-u+X_i)} \int_{0}^{b} C(z)e^{-\alpha z} dz + \left[ 1 - e^{-\alpha (a-u+X_i)} \right] C(0), \ u \in [0,a). \quad (5)
\]

Obviously, the right-hand side of the equation (5) becomes a continuous function, so the solutions of the integral equation (5) is also a continuous function.

**Theorem 3.1** (Banach fixed point theorem)

Let \( H(I) \) be a non-empty and closed set in a Banach space. Assume that \( T : H(I) \to H(I) \) is a contraction mapping, with contraction constant \( q \in [0,1) \), i.e.,

\[
\| T(C_1) - T(C_2) \| \leq q \| C_1 - C_2 \|; \forall C_1, C_2 \in H(I).
\]

Then there exists a unique \( C(.) \in I \) such that \( T(C(u)) = C(u) \), i.e. \( T \) has a unique fixed point in \( H(I) \) (Sofonea et al., 2006).

Now, consider the non-empty and closed set in a Banach space \( (H(I), \| \|_e) \),

where \( H(I) \) is the space of all continuous functions on a compact interval \( I; I = [0, a] \) and \( \| \|_e \) is the sup norm defined as \( \| C \|_e = \sup_{u \in I} |C(u)| \). This norm is also called the supremum norm for all \( u \in [0, a] \) and \( C(.) \in H(I) \) (Venkateshwara et al., 2001).
In this case, let $T$ be an operator in the class of all continuous functions $H(I)$ where $I$ is a compact interval; $I = [0, a]$ and define the operators $T$ by

$$T(C(u)) = 1 + \alpha e^{\alpha(u-a+X_i)} \int_0^b C(z)e^{-\alpha z} dz + (1-e^{-\alpha(a-u-X_i)})C(0).$$  \hspace{1cm} (6)$$

Therefore, the operator $T$ in (6) can be map $H(I)$ into $H(I)$. The following well-known of the Banach fixed point theorem, if the operator $T$ is a contraction, then the fixed point equations $T(C(u)) = C(u)$ have a unique solution (Venkateshwara et al., 2001). To prove the uniqueness of the solution of (6) the following theorem in 3.2 is considered.

**Theorem 3.2** The operator $T$ is the contraction on a metric space $(H(I), \| \|_{\infty})$ with the norm $\|C\|_{\infty} = \sup_{u \in I} |C(u)|$.

**Proof:** To show that $T$ is the contraction and to prove that for all $u \in I$, and two arbitrary function $C_1, C_2 \in H(I)$ in According to (6) one should achieve the following

$$\|T(C_1) - T(C_2)\|_{\infty} = \sup_{u \in I} |C(u)|$$

$$\leq \sup_{u \in I} \left\{ (C_1(0) - C_2(0)) \left(1-e^{-\alpha(a-u-X_i)}\right) \right.$$  
$$+ \alpha e^{\alpha(u-a+X_i)} \int_0^b (C_1(z) - C_2(z)) e^{-\alpha z} dz \right\}.$$  

By Triangular inequality $|C_1(0) - C_2(0)| \leq \sup_{u \in I} |C_1(u) - C_2(u)| = \|C_1 - C_2\|_{\infty}$.

$$= \|C_1 - C_2\|_{\infty} \sup_{u \in I} \left\{ 1 - e^{-\alpha(a-u-X_i)} + \alpha e^{\alpha(u-a+X_i)} \int_0^b e^{-\alpha z} dz \right\}$$
Thus, $\|T(C_1) - T(C_2)\|_\infty = q \|C_1 - C_2\|_\infty$, 

where $q = \left[1 - e^{\alpha X_i - \alpha b}\right] \in [0,1)$, and $q$ is a positive constant.

Therefore, $T$ is the contraction mapping in the non-empty and closed set in a Banach space, with contraction constant $q \in [0,1)$, then there exist a unique of solution such that $T(C(u)) = C(u)$. By Theorem 3.2 and Banach fixed point theorem which was used to guarantee the existence and uniqueness of the solution for ARL.

4. The Solutions of Integral Equation for ARLs

4.1 The explicit formulas

The derived explicit formulas from the solution of integral equation (5) for ARLs are presented as follows:

**Theorem 4.1** The solutions of $T(C(u)) = C(u)$ is 

$$C(u) = e^{ab}(1 + e^{\alpha(a - X_i)} - \alpha b) - e^{\alpha u}, \ u \in [0, a].$$  \hspace{1cm} (7)

**Proof:** According to (5), we have that 

$$C(u) = 1 + \alpha e^{\alpha(a - u + X_i)} \int_0^b C(z) e^{-\alpha z} dz + (1 - e^{-\alpha(a - u - X_i)})C(0), \ u \in [0, a].$$

Let $s = \int_0^b C(z) e^{-\alpha z} dz$. The function $C(u)$ can be rewritten as 

$$C(u) = 1 + \alpha e^{\alpha(u - a + X_i)} s + (1 - e^{-\alpha(a - u - X_i)})C(0).$$ \hspace{1cm} (8)

In particular, if $u = 0$, then $C(0) = 1 + \alpha e^{\alpha(a - a + X_i)} s + (1 - e^{-\alpha(a - a - X_i)})C(0)$. Thus
\[ C(0) = e^{\alpha(a-X_i)} + \alpha s. \]  \hfill (9)

Substituting (9) into (8) then \( C(u) \) as formed

\[ C(u) = 1 + \alpha e^{\alpha(u-a+X_i)} s + (1 - e^{-\alpha(a-u-X_i)}) (e^{\alpha(a-X_i)} + \alpha s). \]

Consequently,

\[ C(u) = 1 + \alpha s + e^{\alpha(a-X_i)} - e^{\alpha u}. \] \hfill (10)

Finding a constant \( s \) from (10) as formed

\[
\begin{align*}
  s &= \int_0^b (1 + \alpha s + e^{\alpha(a-X_i)} - e^{\alpha y}) e^{-\alpha y} dy \\
  &= \int_0^b (1 + \alpha s + e^{\alpha(a-X_i)}) e^{-\alpha y} dy - \int_0^b e^{\alpha y} e^{-\alpha y} dy.
\end{align*}
\]

Here, a constant \( s \) can be rewritten

\[ s = \frac{e^{ab}}{\alpha} (1 - e^{-ab})(1 + e^{\alpha(a-X_i)}) - be^{ab}. \] \hfill (11)

Finally, substituting a constant \( s \) into (10)

\[
\begin{align*}
  C(u) &= 1 + \alpha \left( \frac{e^{ab}}{\alpha} (1 - e^{-ab})(1 + e^{\alpha(a-X_i)}) - be^{ab} \right) + e^{\alpha(a-X_i)} - e^{\alpha u} \\
  &= 1 + e^{ab} (1 - e^{-ab})(1 + e^{\alpha(a-X_i)}) - \alpha be^{ab} + e^{\alpha(a-X_i)} - e^{\alpha u} \\
  &= e^{ab} (1 + e^{\alpha(a-X_i)} - \alpha b) - e^{\alpha u} \quad ; u \geq 0.
\end{align*}
\]

Therefore, the explicit formulas for \( \text{ARL}_0 \) and \( \text{ARL}_1 \) on CUSUM chart can be written:

\[ \text{ARL}_0 = e^{\alpha,b} (1 + e^{\alpha,a(X_i)} - \alpha_0 b) - e^{\alpha,u}, \] \hfill (12)

and
ARL\(_i\) = \(e^{a,b}(1 + e^{(a-X) - \alpha b}) - e^{a,u}\). \hspace{1cm} (13)

### 4.2 Numerical Integral Equation (NIE) Method

This section the authors presents the numerical integral equation (NIE) method to compute the solutions \(C(u) = E_{\alpha}(r_b) < \infty\) of integral equations (5) to extend the function \(C(u)\) into the Fredholm integral equations of the second kind (Wieringa, 1999) as the following form:

\[
C(u) = 1 + \int_0^b C(z)f(z + a - u - X_i)dz + C(0)F(a - u - X_i), \hspace{1cm} (14)
\]

where \(F(u) = 1 - e^{-\alpha u}\) and \(f(u) = \frac{dF(u)}{du} = \alpha e^{-\alpha u}\).

Let \(\tilde{C}(u)\) denote the approximated numerical integral equation (14) using the Gauss-Legendre quadrature rule as follows:

\[
\tilde{C}(a_i) \approx 1 + \sum_{j=1}^{m} w_j \tilde{C}(a_j)f(a_j + a - a_i - X_i) + \tilde{C}(0)F(a - a_i - X_i), \hspace{1cm} (15)
\]

where

\[
i = 1, 2, ..., m, \hspace{0.5cm} w_j \text{ is a weight define different quadrature rules with } w_j = \frac{b}{m} \geq 0,
\]

\[
a_j \text{ is a set of point with } a_j = \frac{b}{m}\left(j - \frac{1}{2}\right); \hspace{0.5cm} j = 1, 2, ..., m.
\]

The previous equation is a system of \(m\) linear equations in the \(m\) unknowns \(\tilde{C}(a_1), \tilde{C}(a_2), ..., \tilde{C}(a_m)\), which can be rearranged as:
\[
\tilde{C}(a_i) = 1 + \sum_{j=2}^{m} w_j \tilde{C}(a_j) f(a_j + a - a_j - X_i) + \tilde{C}(a_i) \left[ F(a - a_i - X_i) + w_1 f(a - X_i) \right]
\]

\[
\tilde{C}(a_2) = 1 + \sum_{j=2}^{m} w_j \tilde{C}(a_j) f(a_j + a - a_2 - X_i) + \tilde{C}(a_i) \left[ F(a - a_2 - X_i) + w_1 f(a + a - a_2 - X_i) \right]
\]

\[\vdots\]

\[
\tilde{C}(a_m) = 1 + \sum_{j=2}^{m} w_j \tilde{C}(a_j) f(a_j + a - a_m - X_i) + \tilde{C}(a_i) \left[ F(a - a_m - X_i) + w_1 f(a_i + a - a_m - X_i) \right].
\]

It can be written in matrix form as:

\[
\mathbf{C}_{m+1} = \mathbf{1}_{m+1} + \mathbf{R}_{m+m} \mathbf{C}_{m+1}\]

where \(\mathbf{C}_{m+1} = \begin{bmatrix} \tilde{C}(a_1) \\ \tilde{C}(a_2) \\ \vdots \\ \tilde{C}(a_m) \end{bmatrix}\), \(\mathbf{1}_{m+1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}\),

\[
\mathbf{R}_{m+m} = \begin{bmatrix} F(a - a_1 - X_i) + w_1 f(a - X_i) & \ldots & w_m f(a_m + a - a_1 - X_i) \\ F(a - a_1 - X_i) + w_1 f(a_1 + a - a_2 - X_i) & \ldots & w_m f(a_m + a + a_2 - X_i) \\ \vdots & & \vdots \\ F(a - a_m - X_i) + w_1 f(a_i + a - a_m - X_i) & \ldots & w_m f(a_m + a - a_m - X_i) \end{bmatrix},
\]

and \(\mathbf{I}_m = \text{diag}(1,1,\ldots,1)\) is identity matrix order \(m\).

Therefore, \(\mathbf{C}_{m+1} = \mathbf{1}_{m+1} + \mathbf{R}_{m+m} \mathbf{C}_{m+1}\), or similarly \((\mathbf{I}_m - \mathbf{R}_{m+m})\mathbf{C}_{m+1} = \mathbf{1}_{m+1}\).

If \((1 - \mathbf{R}_{m+m})\) is invertible and exists, then the unique solution of matrix equation (16) is achieved as follows:

\[
\mathbf{C}_{m+1} = (1 - \mathbf{R}_{m+m})^{-1} \mathbf{1}_{m+1}.
\]

Consequently, the numerical integral equation (NIE) method for ARL on CUSUM chart can be written
\[ \tilde{C}(u) \approx 1 + \sum_{j=1}^{m} w_j \tilde{C}(a_j)f(a_j + a - u - X_j) + \tilde{C}(0)F(a - u - X_j), \]  
\hspace{1cm} (18)

with \( w_j = \frac{b}{m} \) and \( a_j = \frac{b}{m} \left( j - \frac{1}{2} \right) ; j = 1,2,\ldots,m. \)

5. Comparison of Analytical Results

This section derives the explicit formulas and NIE method values for ARL\(_0\) and ARL\(_1\) from equation (12), (13), and (18) with parameters \( a \) and \( b \) for ARL\(_0\) which fixes at 370 and 500. Also, the explicit formulas values are compared with values obtained from the NIE method under these same parameters.

The comparison of efficiency based on percentage of absolute difference (\( \text{Diff} (%) \)) is defined as:

\[ \text{Diff}(\%) = \left| \frac{C(u) - \tilde{C}(u)}{C(u)} \right| \times 100\%. \]  
\hspace{1cm} (19)

where \( C(u) \) is ARLs of the explicit formulas values, and \( \tilde{C}(u) \) is ARLs of the NIE method values.

**Criteria for consideration:** If the \( \text{Diff} (%) \) is less than 2%, then ARL values from the explicit formulas and the NIE method are similar and in good agreement.

From the results in Tables 1-4, the authors applied equation (12), (13), and (18) to evaluate the ARLs for the long memory process on ARFIMA (2,0.2,1) model. The comparison of efficiency between the explicit formulas and NIE method with given \( a = 3, 3.5, \phi_1 = 0.10, -0.10, \phi_2 = 0.20, \) and \( \theta_1 = 0.10 \) for ARL\(_0\) = 370 and 500.
The process in control parameter value ($\alpha_0$) with shift size ($\delta = 0$) had a fixed $\text{ARL}_0 = 370$ and 500. The first row in Tables 1-4 shows that the values of $\text{ARL}_0$ in explicit formulas were close to the NIE method and also approached 370 and 500. The computational CPU time of ARLs by NIE method was computed. The values in parentheses represented the CPU time for calculation with division points, $m = 800$ nodes. The CPU time with the NIE method was about 1.8 - 1.9 hours, this was very high compared to the explicit formulas which equaled less than 1 second.

On the other hand, the process out-of-control was presented parameter value, $\alpha_1 = \alpha_0 (1 + \delta)$ where $\delta = 0.01, 0.03, 0.05, 0.10, 0.20, \text{and} 0.40$. According to the results from Tables 1-4, the percentage of absolute difference of the explicit formulas and NIE method was less than 0.23% calculated using equation (19). In summary, the CPU time of the explicit formulas was less than one second, while the NIE method was approximately at 1.8-1.9 hours.

6. Conclusions

This paper presented the investigation of explicit formulas for the average run lengths of long memory process with the ARFIMA($p,d,q$) on CUSUM control charts with exponential white noise. The accuracy of the proposed explicit formulas in terms of percentage of absolute difference of the explicit formulas and NIE method were checked and compared. The results showed that both methods were similar and in good agreement with the percentage of absolute difference at less than 0.23%. But, the computational CPU time of the explicit formula was less than one second, while the NIE method was approximately 1.8-1.9 hours. Therefore, the explicit formulas are
alternatively preferable to the NIE method because ARL values use a drastically lower computational CPU time.

In conclusion, from the above results, one can see that the explicit formulas and numerical integral equation (NIE) method of ARFIMA$(p,d,q)$ process with exponential white noise on CUSUM control chart can be successfully applied to real world applications for different processes of data, for example in economics, agriculture.

Acknowledgments

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References


Table 1 Comparison of ARL values for ARFIMA(2, 0.2, 1) process using explicit formulas against NIE method when given $a = 3$, $\phi_1 = 0.10$, $\phi_2 = 0.20$ and $\theta_1 = 0.10$, $b = 3.56928$ for $\text{ARL}_0 = 370$ and $b = 3.900538$ for $\text{ARL}_0 = 500$.

<table>
<thead>
<tr>
<th>Shift size ($\delta$)</th>
<th>ARL$_0$ = 370</th>
<th>ARL$_0$ = 500</th>
</tr>
</thead>
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<tr>
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<td>NIE</td>
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<tr>
<td>0.00</td>
<td>370.0004</td>
<td>369.2444 (1.97)</td>
</tr>
<tr>
<td>0.01</td>
<td>347.0438</td>
<td>346.3443 (1.98)</td>
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<tr>
<td>0.03</td>
<td>306.4437</td>
<td>305.8424 (1.84)</td>
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<tr>
<td>0.05</td>
<td>271.8672</td>
<td>271.3478 (1.84)</td>
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<tr>
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<td>205.4008</td>
<td>205.0331 (1.83)</td>
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<tr>
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<td>125.5799 (1.85)</td>
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<td>58.4003</td>
<td>58.3274 (1.82)</td>
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The values in parentheses are CPU times in numerical integration Equation methods (Hours).

Table 2 Comparison of ARL values for ARFIMA(2, 0.2, 1) process using explicit formulas against NIE method when given $a = 3.5$, $\phi_1 = 0.10$, $\phi_2 = 0.20$ and $\theta_1 = 0.10$, $b = 2.9450131$ for $\text{ARL}_0 = 370$ and $b = 3.2604379$ for $\text{ARL}_0 = 500$.

<table>
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<td>62.3003 (1.82)</td>
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The values in parentheses are CPU times in numerical integration Equation methods (Hours).
Table 3: Comparison of ARL values for ARFIMA(2, 0.2, 1) process using explicit formulas against NIE method when given $a = 3$, $\phi_1 = -0.10$, $\phi_2 = 0.20$ and $\theta_1 = 0.10$, $b = 3.390216$ for ARL$_0 = 370$ and $b = 3.715676$ for ARL$_0 = 500$.

<table>
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The values in parentheses are CPU times in numerical integration Equation methods (Hours)

Table 4: Comparison of ARL values for ARFIMA(2, 0.2, 1) process using explicit formulas against NIE method when given $a = 3.5$, $\phi_1 = -0.10$, $\phi_2 = 0.20$ and $\theta_1 = 0.10$, $b = 2.791475$ for ARL$_0 = 370$ and $b = 3.1044675$ for ARL$_0 = 500$.

<table>
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The values in parentheses are CPU times in numerical integration Equation methods (Hours)