**On $(m,n)$-ideals and $(m,n)$-regular ordered semigroups**

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On $(m,n)$-ideals and $(m,n)$-regular ordered semigroups (songkla).tex
On \((m, n)\)-ideals and \((m, n)\)-regular ordered semigroups

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Abstract: Let \(m, n\) be non-negative integers. A subsemigroup \(A\) of an ordered semigroup \((S, \cdot , \leq)\) is called an \((m, n)\)-ideal of \(S\) if 
(i) \(A^m S A^n \subseteq A\), and 
(ii) if \(x \in A, y \in S\) such that 
\(y \leq x\), then \(y \in A\). In this paper, necessary and sufficient conditions for every \((m, n)\)-ideal (resp. \((m, n)\)-quasi-ideal) of an \((m, n)\)-ideal (resp. \((m, n)\)-quasi-ideal) \(A\) of \(S\) is an \((m, n)\)-ideal (resp. \((m, n)\)-quasi-ideal) of \(S\) will be given. Moreover, \((m, n)\)-regularity of \(S\) will be discussed. The results obtained extend the results on semigroups (without order) studied by Bogdanović (1979).

1 Preliminaries

Let \(m, n\) be non-negative integers. A subsemigroup \(A\) of a semigroup \(S\) is called an \((m, n)\)-ideal of \(S\) if

\[A^m S A^n \subseteq A.\]

Here, \(A^0 S = SA^0 = S\). This notion was first introduced and studied by Lajos (1961). Furthermore, the theory of \((m, n)\)-ideals in other structures have also been studied by many authors (see also Akram et al., 2013; Amjad et al., 2014; Lajos, 1963; Yaqoob et al., 2012; Yaqoob et al., 2013; Yaqoob et al., 2014; Yousafzai et al., 2014). A semigroup \(S\) is said

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A semigroup \((S, \cdot)\) together with a partial order \(\leq\) that is compatible with the semigroup operation, meaning that, for any \(a, b, c \in S\),
\[
a \leq b \Rightarrow ac \leq bc, \ ca \leq cb,
\]
is said to be an ordered semigroup (Birkhoff, 1967; Fuchs, 1963). A non-empty subset \(A\) of an ordered semigroup \((S, \cdot, \leq)\) is said to be a subsemigroup of \(S\) if \(ab \in A\) for all \(a, b \in A\) (Kehayopulu, 2006).

If \(A\) and \(B\) are non-empty subsets of an ordered semigroup \((S, \cdot, \leq)\), the set product \(AB\) is defined to be the set of all elements \(ab \in S\) such that \(a \in A\) and \(b \in B\), that is, \(AB = \{ab \mid a \in A, b \in B\}\). And, we write
\[
[A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.
\]

It is observed by Kehayopulu (2006) that the following conditions hold: (1) \(A \subseteq (A]\); (2) \((A)[B] \subseteq (AB]; (3) If \(A \subseteq B\), then \((A] \subseteq (B]; (4) (A \cup B] = (A] \cup (B]; (5) (A \cap B] \subseteq (A] \cap (B].

A non-empty subset \(A\) of an ordered semigroup \((S, \cdot, \leq)\) is called a left (resp. right) ideal of \(S\) if it satisfies the following conditions: (i) \(SA \subseteq A\) (resp. \(AS \subseteq A\)); (ii) \((A] = A\). And, \(A\) is called a two-sided ideal (or simply an ideal) of \(S\) if it is both a left and a right ideal of \(S\) (Kehayopulu, 2006). A subsemigroup \(B\) of \(S\) is called a bi-ideal of \(S\) if (i) \(BSB \subseteq B\); (ii) \((B] = B\) (Kehayopulu, 1992). A non-empty subset \(Q\) of \(S\) is called a quasi-ideal of \(S\) if (i) \((QS] \cap (SQ] \subseteq Q\); (ii) \(Q] = Q\) (Tsingelis, 1991; Kehayopulu, 1994). Note that if \(Q\) is a quasi-ideal of \(S\), then it is a subsemigroup of \(S\). In fact, if \(Q\) is a quasi-ideal of \(S\), then
$QQ \subseteq (QS) \cap (SQ) \subseteq Q$. Finally, a subsemigroup $A$ of $S$ is called an $(m, n)$-ideal of $S$ ($m, n$ are non-negative integers) if (i) $A^mSA^n \subseteq A$; (ii) $A = A$ (Sanborisoot et al., 2012).

We first prove the following theorem.

**Theorem 1.1.** Let $A$ be a non-empty subset of an ordered semigroup $(S, \cdot, \leq)$. Then the intersection of all $(m, n)$-ideals containing $A$ of $S$, denoted by $[A]_{(m,n)}$, is an $(m, n)$-ideal containing $A$ of $S$, and it is of the form

$$[A]_{(m,n)} = \left( \bigcup_{i=1}^{m+n} A^i \cup A^mSA^n \right). \quad (1.1)$$

**Proof.** Let $\{A_i \mid i \in I\}$ be the set of all $(m, n)$-ideals containing $A$ of $S$. Then $\bigcap_{i \in I} A_i$ is a subsemigroup containing $A$ of $S$. For $j \in I$, we have

$$\left( \bigcap_{i \in I} A_i \right)^m S \left( \bigcap_{i \in I} A_i \right)^n \subseteq A_j^mSA_j^n \subseteq A_j.$$

Then $(\bigcap_{i \in I} A_i)^m S (\bigcap_{i \in I} A_i)^n \subseteq \bigcap_{i \in I} A_i$. Since

$$\left( \bigcap_{i \in I} A_i \right) \subseteq \bigcap_{i \in I} (A_i) = \bigcap_{i \in I} A_i \subseteq \left( \bigcap_{i \in I} A_i \right),$$

it follows that $\bigcap_{i \in I} A_i$ is an $(m, n)$-ideal of $S$.

We will show that (1.1) holds. It is easy to see that $(\bigcup_{i=1}^{m+n} A^i \cup A^mSA^n)$ is a subsemigroup of $S$. We now consider:

$$\left( \left( \bigcup_{i=1}^{m+n} A^i \cup A^mSA^n \right) \right)^{m} S$$

$$= \left( \left( \bigcup_{i=1}^{m+n} A^i \cup A^mSA^n \right) \right)^{m-1} \left( \bigcup_{i=1}^{m+n} A^i \cup A^mSA^n \right) S$$

$$\subseteq \left( \left( \bigcup_{i=1}^{m+n} A^i \cup A^mSA^n \right) \right)^{m-1} (AS)$$

$$= \left( \left( \bigcup_{i=1}^{m+n} A^i \cup A^mSA^n \right) \right)^{m-2} \left( \bigcup_{i=1}^{m+n} A^i \cup A^mSA^n \right) (AS)$$
\[
\subseteq \left( \bigcup_{i=1}^{m+n} A^i \cup A^m S A^n \right)^{m-2} (A^2 S] \\
\subseteq (A^m S].
\]

Similarly,
\[
S \left( \left( \bigcup_{i=1}^{m+n} A^i \cup A^m S A^n \right)^n \right) \subseteq (S A^n].
\]

Then,
\[
\left( \left( \bigcup_{i=1}^{m+n} A^i \cup A^m S A^n \right)^m S \left( \bigcup_{i=1}^{m+n} A^i \cup A^m S A^n \right)^n \right) \subseteq (A^m S A^n]
\]
\[
\subseteq \left( \bigcup_{i=1}^{m+n} A^i \cup A^m S A^n \right).
\]

Hence \( \left( \bigcup_{i=1}^{m+n} A^i \cup A^m S A^n \right) \) is an \((m, n)\)-ideal containing \( A \) of \( S \), and
\[
[A]_{(m,n)} \subseteq \left( \bigcup_{i=1}^{m+n} A^i \cup A^m S A^n \right].
\]

Finally, by
\[
(A^m S A^n] \subseteq (\{A\}_{(m,n)})^m S (\{A\}_{(m,n)})^n \subseteq [A]_{(m,n)},
\]

it follows that
\[
\left( \bigcup_{i=1}^{m+n} A^i \cup A^m S A^n \right) \subseteq [A]_{(m,n)}.
\]

This completes the proof. \( \square \)

For an element \( a \) of an ordered semigroup \((S, \cdot, \leq)\), we write \([\{a\}]_{(m,n)}\) (or simply \([a]_{(m,n)}\)) by:
\[
[a]_{(m,n)} = \left( \bigcup_{i=1}^{m+n} \{a\}^i \cup a^m S a^n \right).
\]
To extend the notion of \((m, n)\)-quasi-ideals of semigroups defined by Lajos (1961), we introduce the concept of \((m, n)\)-quasi-ideals of an ordered semigroup \((S, \cdot, \leq)\) as follows: let \(m, n\) be non-negative integers. A subsemigroup \(Q\) of \(S\) is called an \((m, n)\)-quasi-ideal of \(S\) if it satisfies the following conditions:

(i) \(Q^m S \cap (SQ^n) \subseteq Q\);

(ii) \(Q = Q\).

Here, \(Q^0 S = SQ^0 = S\). Note that every \((m, n)\)-quasi-ideal of \(S\) is an \((m, n)\)-ideal of \(S\).

It’s easy to see that if \(Q\) is a quasi-ideal of \(S\), then \(Q\) is an \((m, n)\)-quasi-ideal of \(S\). The following example shows that an \((m, n)\)-quasi-ideal of \(S\) needs not to be a quasi-ideal of \(S\).

**Example 1.1.** Let \(S = \{a, b, c, d\}\) be an ordered semigroup with the multiplication and the order relation are defined by:

\[
\begin{array}{cccc}
 & a & b & c & d \\
\hline 
 a & d & c & d & d \\
b & c & d & d & d \\
c & d & d & d & d \\
d & d & d & d & d \\
\end{array}
\]

\(\leq = \{(a, a), (b, b), (c, c), (d, a), (d, b), (d, c), (d, d)\}\).

We give the covering relation and the figure of \(S\) by:

\(\prec = \{(d, a), (d, b), (d, c)\}\)

Let \(Q = \{a, d\}\). For integers \(m, n > 1\), we obtain that \(Q\) is an \((m, n)\)-quasi-ideal of \(S\) but not a quasi-ideal of \(S\).
As in Theorem 1.1, we have the following.

**Theorem 1.2.** Let \((S, \cdot, \leq)\) be an ordered semigroup. Then the intersection of all \((m, n)\)-quasi-ideals containing a non-empty subset \(A\) of \(S\), denoted by \([A]_{q,(m, n)}\), is an \((m, n)\)-quasi-ideal containing \(A\) of \(S\), and it is of the form

\[
[A]_{q,(m, n)} = \left( \bigcup_{i=1}^{\max\{m,n\}} A^i \right) \cup \left( (A^m S) \cap (SA^n) \right). \tag{1.2}
\]

**Proof.** Let \(\{A_i \mid i \in I\}\) be the set of all \((m, n)\)-quasi-ideals containing \(A\) of \(S\). Then \(\bigcap_{i \in I} A_i\) is a subsemigroup containing \(A\) of \(S\). For \(j \in I\), we have

\[
((\bigcap_{i \in I} A_i)^m S) \cap (S(\bigcap_{i \in I} A_i)^n) \subseteq ((A_j)^m S) \cap (S(A_j)^n) \subseteq A_j
\]

and then \((\bigcap_{i \in I} A_i)^m S) \cap (S(\bigcap_{i \in I} A_i)^n) \subseteq \bigcap_{i \in I} A_i\). Moreover,

\[
\bigcap_{i \in I} A_i \subseteq \bigcap_{i \in I} \bigcap_{i \in I} A_i = \bigcap_{i \in I} A_i \subseteq \bigcap_{i \in I} A_i,
\]

and hence \(\bigcap_{i \in I} A_i\) is an \((m, n)\)-quasi-ideal of \(S\).

Next, we will show that (1.2) holds. Clearly, \(\bigcup_{i=1}^{\max\{m,n\}} A^i \cup ((A^m S) \cap (SA^n)) \neq \emptyset\). Let \(x, y \in \bigcup_{i=1}^{\max\{m,n\}} A^i \cup ((A^m S) \cap (SA^n))\). If \(x \in (A^m S) \cap (SA^n)\) or \(y \in (A^m S) \cap (SA^n)\), then

\[
xy \in (A^m S) \cap (SA^n) \subseteq \left( \bigcup_{i=1}^{\max\{m,n\}} A^i \right) \cup \left( (A^m S) \cap (SA^n) \right).
\]

Let \(x, y \in \bigcup_{i=1}^{\max\{m,n\}} A^i\); then there exist \(j, k\) in \(\{1, 2, \ldots, \max\{m, n\}\}\) such that \(x \in (A^j)\) and \(y \in (A^k)\). If \(1 < j + k \leq \max\{m, n\}\), then

\[
xy \in \left( \bigcup_{i=1}^{\max\{m,n\}} A^i \right) \subseteq \left( \bigcup_{i=1}^{\max\{m,n\}} A^i \right) \cup \left( (A^m S) \cap (SA^n) \right).
\]

If \(\max\{m, n\} < j + k\), then \(m, n < j + k\), that is, \((A^{i+k}) = (A^{m+(j+k-m)}) \subseteq (A^m S)\) and
\[(A^{j+k}) = (A^{(j+k-n)+n}) \subseteq (SA^n).\] Hence

\[xy \in (A^{j+k}) \subseteq (A^m S \cap SA^n) \subseteq \left( \bigcup_{i=1}^{\max\{m,n\}} A^i \right) \cup \left( (A^m S \cap SA^n) \right) \subseteq (S^n).\]

This shows that \(\left( \bigcup_{i=1}^{m+n} A^i \right) \cup (A^m S \cap SA^n)\) is a subsemigroup of \(S\). We now consider:

\[\left( \bigcup_{i=1}^{\max\{m,n\}} A^i \right) \cup \left( \left( A^m S \cap (SA^n) \right)^m \right) \subseteq \left( \bigcup_{i=1}^{\max\{m,n\}} A^i \right) \cup \left( \left( A^m S \right)^m \right) \subseteq \left( \bigcup_{i=1}^{\max\{m,n\}} A^i \right) \cup \left( \left( A^m S \right)^{m-1} \right).\]

Similarly,

\[S^n \left( \left( \bigcup_{i=1}^{\max\{m,n\}} A^i \right) \cup \left( \left( A^m S \cap (SA^n) \right)^n \right) \right) \subseteq (SA^n).\]

Then,

\[\left( Q^m S \right) \cap \left( SQ^n \right) \subseteq (A^m S \cap (SA^n)) \subseteq \left( \bigcup_{i=1}^{\max\{m,n\}} A^i \right) \cup \left( \left( A^m S \cap (SA^n) \right) \right),\]
where \( Q = \left( \bigcup_{i=1}^{\max\{m,n\}} A^i \right) \cup \left( (A^m S) \cap (SA^n) \right) \). Now, 
\[
\left( \left( \bigcup_{i=1}^{\max\{m,n\}} A^i \right) \cup \left( (A^m S) \cap (SA^n) \right) \right) = \left( \left( \bigcup_{i=1}^{\max\{m,n\}} A^i \right) \cup \left( (A^m S) \cap (SA^n) \right) \right) \subseteq \left( \bigcup_{i=1}^{\max\{m,n\}} A^i \right) \cup \left( (A^m S) \cap (SA^n) \right).
\]
Thus \( \left( \bigcup_{i=1}^{\max\{m,n\}} A^i \right) \cup \left( (A^m S) \cap (SA^n) \right) \) is an \((m, n)\)-quasi-ideal containing \( A \) of \( S \), and 
\[ [A]_{q,(m,n)} \subseteq \left( \bigcup_{i=1}^{\max\{m,n\}} A^i \right) \cup \left( (A^m S) \cap (SA^n) \right). \]
By 
\[
\left( \bigcup_{i=1}^{\max\{m,n\}} A^i \right) \subseteq \left( [A]_{q,(m,n)} \cup \ldots \cup [A]_{q,(m,n)}^{\max\{m,n\}} \right) \subseteq [A]_{q,(m,n)}
\]
and 
\[
(A^m S) \cap (SA^n) \subseteq \left( ([A]_{q,(m,n)})^m S \right) \cap \left( S([A]_{q,(m,n)})^n \right) \subseteq [A]_{q,(m,n)},
\]
it follows that 
\[
\left( \bigcup_{i=1}^{\max\{m,n\}} A^i \right) \cup \left( (A^m S) \cap (SA^n) \right) \subseteq [A]_{q,(m,n)}. \]
This shows that (1.2) holds, and the proof is completed.

For an element \( a \) of an ordered semigroup \((S, \cdot, \leq)\), we write \([\{a\}]_{q,(m,n)}\) (or simply \([a]_{q,(m,n)}\)) by 
\[
[a]_{q,(m,n)} = \left( \bigcup_{i=1}^{\max\{m,n\}} \{a\}^i \right) \cup \left( (a^m S) \cap (Sa^n) \right).
\]

In closing this section we quote the following two results proved by Sanborisoot et al. (2012).
Lemma 1.1. The following conditions hold for an ordered semigroup $(S, \cdot, \leq)$ and $a \in S$:

(1) $([a]_{(m,0)})^m S \subseteq (a^m S]$ for any positive integer $m$.

(2) $S([a]_{(0,n)})^n \subseteq (Sa^n]$ for any positive integer $n$.

(3) $([a]_{(m,n)})^m S([a]_{(m,n)})^n \subseteq (a^m Sa^n]$ for any positive integers $m, n$.

Theorem 1.3. Let $(S, \cdot, \leq)$ be an ordered semigroup. Let $m, n$ be positive integers. Let $\mathcal{R}_{(m,0)}$ be the set of all $(m,0)$-ideals of $S$, and let $\mathcal{L}_{(0,n)}$ be the set of all $(0,n)$-ideals of $S$. Then the following conditions hold:

(1) $S$ is $(m,0)$-regular if and only if for all $R \in \mathcal{R}_{(m,0)}$, $R = (R^m S]$.

(2) $S$ is $(0,n)$-regular if and only if for all $L \in \mathcal{L}_{(0,n)}$, $L = (SL^n]$.

2 Main Results

Let $A$ be a subsemigroup of an ordered semigroup $(S, \cdot, \leq)$. For a non-empty subset $B$ of $A$, we let

$$(B]_A = \{y \in A \mid y \leq b \text{ for some } b \in B\}.$$ 

It is clear that $(B]_A \subseteq (B]$, and the equality holds in the following lemma.

Lemma 2.1. If $A$ is an $(m,n)$-ideal of an ordered semigroup $(S, \cdot, \leq)$, then $(B]_A = (B]$ for any non-empty subset $B$ of $A$.

Lemma 2.2. Let $A$ be an $(m,n)$-ideal of an ordered semigroup $(S, \cdot, \leq)$, and let $\emptyset \neq B \subseteq A$. Then

$$(([B_A]_{(m,n)})^m S([B_A]_{(m,n)})^n] = (B^m S B^n]$$

where $[B_A]_{(m,n)} = \left(\bigcup_{i=1}^{m+n} B^i \cup B^m A B^n\right)_A$. 
Proof. We have

\[
\left( \bigcup_{i=1}^{m+n} B^i \cup B^m AB^n \right)^m S \left( \bigcup_{i=1}^{m+n} B^i \cup B^m AB^n \right)^n_A \\
\subseteq \left( \left( \bigcup_{i=1}^{m+n} B^i \cup B^m AB^n \right)^m S \left( \bigcup_{i=1}^{m+n} B^i \cup B^m AB^n \right)^n \right)_A \\
\subseteq \left( \left( \bigcup_{i=1}^{m+n} B^i \cup B^m AB^n \right)^m \left( \bigcup_{i=1}^{m+n} B^i \cup B^m AB^n \right)^n \right)_A \\
= \left( \bigcup_{i=1}^{m+n} B^i \cup B^m AB^n \right)^m \left( \bigcup_{i=1}^{m+n} B^i \cup B^m AB^n \right)^n.
\]

Let \( x \in ([B_A]_{(m,n)})^m S ([B_A]_{(m,n)})^n \). Then \( x \leq y^m s z^n \) for some \( s \in S \) and \( y, z \in \bigcup_{i=1}^{m+n} B^i \cup B^m AB^n \). If \( y, z \in \bigcup_{i=1}^{m+n} B^i \), then \( y \in B^p \), \( z \in B^q \) for some \( p, q \in \{1,2,\ldots,m+n\} \); hence \( x \in \left( B^{mp} S B^{nq} \right) \subseteq \left( B^m S B^n \right) \). If \( y \in \bigcup_{i=1}^{m+n} B^i \), \( z \in B^m S B^n \), then \( y \in B^p \) for some \( p \in \{1,2,\ldots,m+n\} \); hence \( x \in \left( B^{mp} S B^{nq} \right) \subseteq \left( B^m S B^n \right) \). If \( y \in B^m S B^n \), \( z \in \bigcup_{i=1}^{m+n} B^i \), then \( z \in B^q \) for some \( q \in \{1,2,\ldots,m+n\} \); hence \( x \in \left( (B^m S B^n)^m S B^{nq} \right) \subseteq \left( B^m S B^n \right) \). Finally, if \( y, z \in B^m S B^n \), then \( x \in ((B^m S B^n)^m S (B^m S B^n)^n) \subseteq (B^m S B^n) \). This shows that

\[ ([B_A]_{(m,n)})^m S ([B_A]_{(m,n)})^n \subseteq (B^m S B^n). \]

By

\[ (B^m S B^n) \subseteq (([B_A]_{(m,n)})^m S ([B_A]_{(m,n)})^n), \]

it follows that

\[ ([B_A]_{(m,n)})^m S ([B_A]_{(m,n)})^n \] = \( (B^m S B^n) \),

as required. This completes the proof. \( \square \)

Theorem 2.1. Let \( A \) be an \((m,n)\)-ideal of an ordered semigroup \((S, \cdot, \leq)\). Then every
(m, n)-ideal of A is an (m, n)-ideal of S if and only if for each non-empty subset B of A,

\[ B^m SB^n \subseteq [B_A]_{(m,n)} \]  \hspace{1cm} (2.1)

where \([B_A]_{(m,n)} = \left( \bigcup_{i=1}^{m+n} B^i \cup B^m AB^n \right)_A \).

Proof. Assume first that every (m, n)-ideal of A is an (m, n)-ideal of S. Let \( \emptyset \neq B \subseteq A \). Since \([B_A]_{(m,n)} \) is an (m, n)-ideal of A, it follows by assumption that \([B_A]_{(m,n)} \) is an (m, n)-ideal of S. By Lemma 2.2,

\[ B^m SB^n \subseteq (B^m SB^n) = ([B_A]_{(m,n)})^m S([B_A]_{(m,n)})^n \subseteq ([B_A]_{(m,n)}) = [B_A]_{(m,n)}. \]

Conversely, we assume that the equation (2.1) holds for any non-empty subset of A. Let C be an (m, n)-ideal of A. Then \( C \subseteq A \) and

\[ C^m SC^n \subseteq (C \cup C^2 \cup \ldots \cup C^{m+n} \cup AC^n)_A \subseteq (C)_A = C. \]

By Lemma 2.1, \((C) = C\). Therefore, C is an (m, n)-ideal of S. \( \square \)

For \( m = 0, n = 1 \) (resp. \( m = 1, n = 0 \)), we have the following corollary:

**Corollary 2.1.** Let A be a left (resp. right) ideal of an ordered semigroup \((S, \cdot, \leq)\). Then every left (resp. right) ideal of A is a left (resp. right) ideal of S if and only if for each non-empty subset B of A,

\[ SB \subseteq (B \cup AB)_A \hspace{0.5cm} (\text{resp.} \hspace{0.5cm} BS \subseteq (B \cup BA)_A). \]

Moreover we have the following, taking \( m = 1, n = 1 \):

**Corollary 2.2.** Let A be a bi-ideal of an ordered semigroup \((S, \cdot, \leq)\). Then every bi-ideal of A is a bi-ideal of S if and only if for each non-empty subset B of A,

\[ BSB \subseteq (B \cup B^2 \cup BAB)_A. \]
Example 2.1. Let $S = \{a, b, c, d\}$ be an ordered semigroup with the multiplication and the order relation are defined by:

\[
\begin{array}{c|cccc}
  & a & b & c & d \\
\hline
  a & a & a & a & a \\
  b & a & a & a & a \\
  c & a & a & b & a \\
  d & a & a & b & b \\
\end{array}
\]

\[\leq = \{(a, a), (a, b), (b, b), (c, c), (d, d)\} \]

We give the covering relation and the figure of $S$ by:

\[\prec = \{(a, b)\}\]

Then $A = \{a, d\}$ is a bi-ideal of $S$, and $\{a\}$ is a bi-ideal of $A$. It is easy to verify that, for each non-empty subset $B$ of $A$, we have $BSB \subseteq (B \cup B^2 \cup BAB)_{|A}$. Thus, by Corollary 2.2, $\{a\}$ is a bi-ideal of $S$.

Theorem 2.2. Let $Q$ be an $(m, n)$-quasi-ideal of an ordered semigroup $(S, \cdot, \leq)$. Then every $(m, n)$-quasi-ideal of $Q$ is an $(m, n)$-quasi-ideal of $S$ if and only if for each non-empty subset $D$ of $Q$,

\[
(D^m S) \cap (SD^n) \subseteq [D_Q]_{|Q, (m, n)}
\]  

where $[D_Q]_{|Q, (m, n)} = \left( \bigcup_{i=1}^{\max\{m, n\}} D^i \right)_Q \cup \left( \left( D^m Q \right)_Q \cap \left( QD^n \right)_Q \right)$.

Proof. Assume that every $(m, n)$-quasi-ideal of $Q$ is an $(m, n)$-quasi-ideal of $S$. If $D \subseteq Q$ is
non-empty, then, by Theorem 1.2, \([D_Q]_{q,(m,n)}\) is an \((m, n)\)-quasi-ideal of \(Q\). By assumption,

\[
(D^m S) \cap (SD^n) \subseteq ([D_Q]_{q,(m,n)})^m S \cap (S([D_Q]_{q,(m,n)}))^n) \subseteq [D_Q]_{q,(m,n)}.
\]

Conversely, we assume that the equation (2.2) holds for any non-empty subset of \(Q\). Let \(C\) be an \((m, n)\)-quasi-ideal of \(Q\). Then \(C \subseteq Q\) and

\[
(C^m S) \cap (SC^n) \subseteq [C]_{Q,q,(m,n)} = C.
\]

By Lemma 2.1, \((C) = C\). Therefore, \(C\) is an \((m, n)\)-quasi-ideal of \(S\).

\[
\square
\]

For \(m = 1, n = 1\), we have the following corollary:

**Corollary 2.3.** Let \(Q\) be a quasi-ideal of an ordered semigroup \((S, \cdot, \leq)\). Then every quasi-ideal of \(Q\) is a quasi-ideal of \(S\) if and only if for each non-empty subset \(D\) of \(Q\),

\[
(DS) \cap (SD) \subseteq (D)_{Q} \cup ((DQ)_{Q} \cap (QD)_{Q}).
\]

**Example 2.2.** Let \(S = \{a, b, c, d, f\}\) be an ordered semigroup with the multiplication and the order relation are defined by:

\[
\begin{array}{c|cccc}
\cdot & a & b & c & d & f \\
\hline
a & a & a & a & a & a \\
b & a & b & a & d & a \\
c & a & f & c & c & f \\
d & a & b & d & d & b \\
f & a & f & a & c & a \\
\end{array}
\]

\[\leq = \{(a, a), (a, b), (a, c), (a, d), (a, f), (b, b), (c, c), (d, d), (f, f)\}\].

We give the covering relation and the figure of \(S\) by:

\[\prec = \{(a, b), (a, c), (a, d), (a, f)\}\]
Then \( Q = \{a, c, f\} \) is a quasi-ideal of \( S \), and the quasi-ideals of \( Q \) are \( D_1 = \{a\}, D_2 = \{a, c\} \) and \( D_3 = \{a, f\} \). For each non-empty subset \( C \) of \( Q \), we have \((CS)\cap(SC) \subseteq (C)_{Q} \cup ((CQ)_{Q} \cap (QC)_{Q})\). By Corollary 2.3, \( D_1, D_2, D_3 \) are quasi-ideals of \( S \).

**Example 2.3.** Let \( S = \{a, b, c, d, f\} \) be an ordered semigroup with the multiplication and the order relation are defined by:

\[
\begin{array}{c|ccccc}
   & a & b & c & d & f \\
\hline
a & d & b & b & d & f \\
b & b & b & b & b & f \\
c & b & b & c & b & f \\
d & d & d & b & d & f \\
f & b & b & f & b & f \\
\end{array}
\]

\[
\leq = \{(a, a), (a, b), (a, c), (a, f), (b, b), (b, c), (b, f), (c, c), (d, d), (d, b), (d, c), (d, f), (f, f)\}.
\]

We give the covering relation and the figure of \( S \) by:

\[
\prec = \{(a, b), (b, c), (b, f), (d, b)\}
\]
It is easy to verify that $Q = \{a, b, d\}$ is an $(m, n)$-quasi-ideal of $S$ for any integers $m, n \geq 2$, and the $(m, n)$-quasi-ideal of $Q$ is $\{b, d\}$. For each non-empty subset $C$ of $Q$, we have $(C^m S) \cap (S C^n) \subseteq [C_Q]_{q,m,n}$. By Theorem 2.2, $\{b, d\}$ is also a quasi-ideal of $S$.

Let $(S, \cdot, \leq)$ be an ordered semigroup, and let $m, n$ be non-negative integers. Then $S$ is said to be $(m, n)$-regular (Sanborisoot et al., 2012), if for any $a$ in $S$ there exists $x$ in $S$ such that $a \leq a^m x a^n$, that is, if $a \in (a^m Sa^n)$.

**Example 2.4.** Let $S = \{a, b, c, d\}$ be an ordered semigroup with the multiplication and the order relation are defined by:

<table>
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<th>b</th>
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<td>d</td>
<td>c</td>
<td>d</td>
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<td>d</td>
</tr>
</tbody>
</table>

$\leq = \{(a,a), (a,b), (a,c), (a,d), (b,b), (c,c), (c,d), (d,d)\}$.

We give the covering relation and the figure of $S$ by:

$\preceq = \{(a,b), (a,c), (c,d)\}$

Then $S$ is $(m, n)$-regular for any integer $m, n \geq 1$.

**Theorem 2.3.** Let $(S, \cdot, \leq)$ be an ordered semigroup. Then $S$ is $(m, n)$-regular if and only
if

$$\forall R \in \mathcal{R}_{(m,0)}, \forall L \in \mathcal{L}_{(0,n)}, R \cap L = (R^m L^n)$$

(2.3)

where \( \mathcal{R}_{(m,0)} \) is the set of all \((m, 0)\)-ideals of \( S \) and \( \mathcal{L}_{(0,n)} \) is the set of all \((0, n)\)-ideals of \( S \).

**Proof.** The assertion is obvious if \( m = 0, n = 0 \). If \( m = 0, n \neq 0 \), we have to show that \( S \) is \((0, n)\)-regular if and only if \( \forall L \in \mathcal{L}_{(0,n)}, L = (SL^n) \), and this follows by Theorem 1.3 (2).

Similarly, for \( m \neq 0, n = 0 \). This is obtained by Theorem 1.3 (1).

Finally, we let \( m \neq 0, n \neq 0 \). Assume that \( S \) is \((m, n)\)-regular. Let \( R \in \mathcal{R}_{(m,0)} \) and \( L \in \mathcal{L}_{(0,n)} \). We have \( (R^m L^n) \subseteq R \cap L \). Let \( a \in R \cap L \). Since \( S \) is regular, there exists \( x \) in \( S \) such that \( a \leq a^m xa^n \). We have

\[
\begin{align*}
a \leq a^m xa^n \\
\leq a^{2m-1} xa^n xa^n \\
\leq a^{3m-2} xa^n xa^n xa^n \\
\vdots \\
\leq a^{nm-(n-1)}(xa^n)^n \\
\in R^{nm-(n-1)}L^n \\
\subseteq R^m L^n \\
\subseteq (R^m L^n).
\end{align*}
\]

Thus \( R \cap L \subseteq (R^m L^n) \).

Conversely, we assume that (2.3) holds. Let \( a \in S \). Since \([a]_{(m,0)} \in \mathcal{R}_{(m,0)} \) and \( S \in \mathcal{L}_{(0,n)} \), we have

\[
[a]_{(m,0)} = [a]_{(m,0)} \cap S = ([a]_{(m,0)})^m S^n \subseteq (([a]_{(m,0)})^m S^n).
\]
By Lemma 1.1, $[a]_{(m,0)} \subseteq (a^m S]$. Similarly, $[a]_{(0,n)} \subseteq (Sa^n]$. From

$$
a \in [a]_{(m,0)} \cap [a]_{(0,n)}
\subseteq (a^m S] \cap (Sa^n]
= ((a^m S])^m ((Sa^n])^n
\subseteq (a^m S][Sa^n]
\subseteq (a^m Sa^n],
$$

we conclude that $S$ is $(m, n)$-regular. We now complete the proof.

\[\square\]

**Corollary 2.4.** Let $(S, \cdot, \leq)$ be an ordered semigroup. Then $S$ is $(m, n)$-regular if and only if

$$\forall a \in S, [a]_{(m,0)} \cap [a]_{(0,n)} = ([a]_{(m,0)})^m ([a]_{(0,n)})^n].$$

**Theorem 2.4.** Let $(S, \cdot, \leq)$ be an ordered semigroup. Then $S$ is $(m, n)$-regular if and only if

$$\forall a \in S, [a]_{(m,n)} = (a^m Sa^n].$$

**Proof.** Assume that $S$ is $(m, n)$-regular. Let $a \in S$ and $x \in [a]_{(m,n)}$. Then, by Theorem 1.1, $x \leq y$ for some $y$ in $\bigcup_{i=1}^{m+n} a^i \cup a^m Sa^n$. If $y \in a^m Sa^n$, we are done. Suppose that $y \in \bigcup_{i=1}^{m+n} a^i$; then $y = a^p$ for some $p \in \{1, 2, \ldots, m + n\}$. We have

$$x \in (a^p] \subseteq ((a^m Sa^n]^p] \subseteq ((a^m Sa^n] = (a^m Sa^n].$$

Since $(a^m Sa^n] \subseteq [a]_{(m,n)}, [a]_{(m,n)} = (a^m Sa^n].$

Conversely, if $a \in S$, then $a \in [a]_{(m,n)} = (a^m Sa^n]$, and hence $S$ is $(m, n)$-regular.  

\[\square\]

**Example 2.5.** We consider the ordered semigroup which is defined in Example 2.2. We have
\[a_{(1,1)} = (a), \quad b_{(1,1)} = (\{a, b\}), \quad c_{(1,1)} = (\{a, c\}), \quad d_{(1,1)} = (\{a, d\}), \quad \text{and} \quad f_{(1,1)} = (\{a, f\}).\]

Then, by Theorem 2.4, \(S\) is regular.

References


Bogdanović, S. 1979. \((m, n)\)-ideaux et les demi-groupes \((m, n)\)-reguliers. Review of Research. Faculty of Science. Mathematics Series. 9, 169-173.


