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Abstract: In this paper, we investigate the new type of retrial queueing model with feedback and working breakdown services. The regular busy server may become defective by disasters (negative customers) at any point of time. The negative customers only arrive at service time of positive customer and will remove the positive customer from the service. At a failure instant, the main server is sent to the repair and the repair period immediately begins. During the repair period, the server gives service in lower speed (called working breakdown period). The steady state probability generating function for system size and orbit size are obtained by using the method of supplementary variable technique. We also obtain some analytic expressions for various performance measures such as system state probabilities, mean orbit size, and mean system size of this model and some important special cases are discussed. Finally, some numerical examples are presented to study the impact of the system parameters.

Keywords: retrial queue; $G$ queue; feedback; working breakdown services;

1. Introduction

The topic of the retrial queues in queueing theory has been interested research topic in the last two decades. The concept of retrial queues has a great efforts and interest by many researchers (Artalejo, 2010; Artalejo and Gomez-Corral, 2008). Such a queueing models are surely arise applications in the performance analysis of a wide range of systems in data distributed networks, telecommunications, traffic management on high-speed networks and production engineering.

The concept of negative customers (called $G$-queues) first developed by Gelenbe (1989) in computers, neural networks and communication networks. The named $G$-queue (negative
customers) has been adopted for the queue with negative customers in the acknowledgment of Gelenbe. The negative customers (disasters) arrive only at the regular service time of the positive customers (ordinary customers). Negative customers cannot accumulate in a queue and do not receive service, and will remove the positive customers being in service from the system. These types of negative customers cause the server breakdown and the service channel will fail for a short interval of time. At a failure instant, the main server is sent to the repair and the repair period immediately begins. The repaired server is assumed to be as good as a new server. Tan Van Do (2011) has presented a survey on queueing systems with $G$-networks, negative customers and applications. Further, such models are motivated by recent advanced applications in computer systems and data communication networks. Recently, Kim and Lee (2014) have discussed queueing models with breakdowns and repairs.

Queueing models with different service rates were studied by various authors in the past. The initiative of these models almost made the change of the service rate depending on the situation of the system. Such as queues in random environment, queues with breakdown and working breakdown or models with vacations and working vacations. Servi and Finn (2002) introduced an M/M/1 queueing system with working vacations. Wu and Takagi (2006) extended the M/M/1/WV queue to an M/G/1/WV queue. Authors like Arivudainambi et al. (2014), Gao et al. (2014), Rajadurai et al. (2016), Zhang and Hou (2012), Zhang and Liu (2015) and Rajadurai (2018a, 2018b) analyzed queueing systems with working vacations.

The concept of the working breakdowns was first introduced by Kalidass and Ramanath (2012). That is, the system may become defective by disasters at any point of time when the regular busy server is in operation, the system should be ready with a substitute (standby) server in preparation for possible main server failures. The substitute server renders services to
customers while the main server is repaired. The service rate of the substitute server is different from (lower than) that of the main server. At the instant of the repair completion, the main server returns to the system and becomes available. Additionally, the working breakdown service can decrease complaints from the customers who should wait for the main server to be, repaired and reduces the cost of waiting customers. Therefore, the working breakdown service is a more reasonable repair policy for unreliable queueing systems. Recently, Kim and Lee (2014) have discussed a model M/G/1 queueing system with disasters and working breakdown services.

Motivated by this factor, in this work, a new class of M/G/1 retrial queue with negative customers, feedback under working breakdown services and working vacation services is introduced. During the period of working vacation and working breakdown, the server works in different rate of services. The analytical results of this model are very useful and helpful for decision makers for designing a management policy. This model has a potential application in the telephone consultation of medical service systems, stochastic production and inventory systems with a multipurpose production facility and the machine replacement problems. The rest of this work has organized as follows. The mathematical model description of this work is described in section 2. The steady state governing equations, the number of customers in the orbit for different states are obtained in section 3. In section 4, some important system performance measures are given. In section 5, we analyze some special cases of our model, which are consistent with the existing literature. The numerical examples are presented for various parameters on the system performance and cost optimization are analysed in section 6. Finally, conclusions of the work are given in section 7.
2. Description of the model

We investigate an M/G/1 retrial $G$-queue with feedback under working vacations and working breakdowns (M/G/1/WB). The basic assumption of this model is described as follows:

- **The arrival process**: There are two types of customers arriving into the system: ordinary customers (positive customers) and disasters (negative customers). Assume that both types of customers arrive from outside the system according to independent Poisson processes with rates $\lambda$ and $\delta$ respectively.

- **The retrial process**: If an arriving positive customer finds that the server is free, the customer begins his service immediately. Otherwise an arriving customer finds the server busy or lower speed service, the arrivals join the pool of blocked customers called an orbit in accordance with FCFS discipline, which means that only one customer at the head of the orbit queue is allowed access to the server. Inter-retrial times have an arbitrary distribution $R(x)$ with corresponding Laplace Stieltjes Transform (LST) $R'(\varphi)$.

- **The regular service process**: Whenever a new positive customer or retry positive customer arrives at the server idle state then the server immediately starts normal service for the arrivals. The service time has a general distribution and it is denoted by the random variable $S$ with distribution function (d.f) $S(x)$ having LST $S'(\varphi)$.

- **The feedback rule**: After completion of service for each customer, the unsatisfied customers may rejoin into the orbit as a feedback customer for receiving another service with probability $p$ ($0 \leq p \leq 1$) or may leave the system with complement probability $q$ ($=1-p$).

- **The removal rule and the working breakdown process**: The negative customers (disasters) arrive only at the regular service time of the positive customers. Negative customers cannot accumulate in a queue and do not receive service, and will remove the positive customers.
being in service from the system. These types of negative customers cause the server breakdown and the service channel will fail for a short interval of time. At a failure instant, the main server is sent to the repair and the repair period immediately begins. The repair time follows an exponential distribution with rate of $\eta$. The repaired server is assumed to be as good as a new server. However, disaster occurs in the regular busy server, the server goes for a working breakdown. During working breakdown period, the substitute server works in lower service rate to arriving customers ($\mu_w < \mu$). When a repair ends, if there are customers in the orbit, the server switches to the normal working level and will start a new busy period. Otherwise, it is idle and ready for serving new arrivals. During the working breakdown periods (lower speed services), the service time follows a general random variable $S_w$ with distribution function $S_w(t)$ and LST $S_w^*(\theta)$.

- **The multiple working vacations process:** The server begins a working vacation each time when the orbit becomes empty and the vacation time follows an exponential distribution with parameter $\theta$. During a vacation period if any customer arrives, the server gives service at a lower speed service rate ($\mu_w < \mu$). If any customers in the orbit at a lower speed service completion instant in the vacation period, the server will stop the vacation and come back to the normal busy period which means vacation interruption happens. Otherwise, it continues the vacation. When a vacation ends, if there are customers in the orbit, the server switches to the normal working level. Otherwise, the server begins another vacation. During the working vacation periods (lower speed services), the service time follows a general random variable $S_w$ with distribution function $S_w(t)$ and LST $S_w^*(\theta)$.

- **The lower speed service process:** we consider the working vacation period and working breakdown period as the lower speed service period
• We assume that all the random variables (inter-arrival times, retrial times, regular service times and lower speed service times) defined above are independent of each other.

2.1 Practical justifications of the suggested model

The suggested model has practical real life application in the telephone consultation of medical service systems. Nowadays, many doctors have opened telephone consultation services to patients (called positive customers). Here, we consider a telephone consultation service system staffed with a chief physician (main server) and a physician assistant (substitute server or working breakdown server). The physician assistant only provides service to the patients when the chief physician is on vacation (working vacation) and the service rate of the physician assistant is usually slower than the chief physician. In generally, there is a phone operator who is responsible to establish communications between doctors and patients or notes down the order of the calls, corresponding to the ‘orbit’. If the line is busy when a patient makes a call, he cannot queue but tries again sometime later (retrial), otherwise he is served immediately by the chief physician or the physician assistant. During the patients’ consultation time, the telephone signal status is very low or no network coverage (negative customer), and the patient’s call has lost service. Once the signal strength is full (repaired), then the system is again treated as good as new to serve.

When the chief physician finds no patient call, he will need to rest from his work, i.e., go on a vacation. During the chief physician’s vacation period, the physician assistant will serve the patients, if any, and after his service completion if there are patients in the system, the chief physician will come back from his vacation whether his vacation has ended or not, i.e., vacation interruption happens. Meanwhile, if there is no patient when a vacation ends, the chief physician begins another work vacation (multiple working vacations), otherwise, the chief physician takes
over as the physician assistant. To understand the patient’s condition, the chief physician will restart his service no matter how long the physician assistant has served the patient. On the other hand, to minimize the idle time of the chief physician, immediately on a service completion, the phone operator will call (or search for) the customers who are in orbit under FCFS and the search time is assumed to be generally distributed, which is corresponding to the general retrial time policy.

2.2. Notations and probabilities

In steady state, we assume that $R(0)=0$, $R(\infty)=1$, $S(0)=0$, $S(\infty)=1$, $S_\mathrm{w}(0)=0$, $S_\mathrm{w}(\infty)=1$ are continuous at $x=0$. The following notations and probabilities are used in sequent sections:

- $r(x) =$ the hazard rate (conditional completion rate) for retrial of $R(x)$; i.e., $\theta(x)dx = \frac{dR(x)}{1-R(x)}$.
- $\mu(x) =$ the hazard rate for service of $S(x)$; i.e., $\mu(x)dx = \frac{dS(x)}{1-S(x)}$.
- $\mu_\mathrm{w}(x) =$ the hazard rate for lower rate service of $S_\mathrm{w}(x)$; i.e., $\mu_\mathrm{w}(x)dx = \frac{dS_\mathrm{w}(x)}{1-S_\mathrm{w}(x)}$.
- $N(t) =$ the number of customers in the orbit at time $t$.
- $C(t) =$ the state of the server at time $t$.
- $R(t) =$ the elapsed retrial time.
- $S^i(t) =$ the elapsed service time on $i^{th}$ phase.
- $S^\mathrm{w}_\mathrm{w}(t) =$ the elapsed lower rate service time.
- $P_\emptyset(t) =$ the probability that the system is empty at time $t$.
- $W_\emptyset(t) =$ the probability that the system is empty at time $t$ and the server is in working vacation and breakdown (lower speed service).
- $R_n(x,t) =$ the probability that at time $t$ there are exactly $n$ customers in the orbit with the elapsed retrial time of the test customer undergoing retrial lying in between $x$ and $x+dx$.
- $\Pi_n(x,t) =$ the probability that at time $t$ there are exactly $n$ customers in the orbit with the elapsed normal service time of the test customer undergoing service lying in between $x$ and $x+dx$. 
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- \( W_n(x,t) \) is the probability that at time \( t \) there are exactly \( n \) customers in the orbit with the elapsed lower rate service time of the test customer undergoing service lying in between \( x \) and \( x+dx \).

3. Steady state analysis

For an M/G/1 retrial G-queue with feedback under working vacations and working breakdowns (M/G/1/WVB), we develop the steady state difference-differential equations based on supplementary variable method.

For further development of this retrial queueing model, let us define the random variable

\[
C(t) = \begin{cases} 
0, & \text{if the server is free and in working vacation and working breakdown period}, \\
1, & \text{if the server is free and in regular service period}, \\
2, & \text{if the server is busy and in regular service period on both phases at time } t, \\
3, & \text{if the server is busy and in lower speed service period period at time } t.
\end{cases}
\]

Thus the supplementary variables are introduced in order to obtain a bivariate Markov process \( \{C(t), N(t); t \geq 0\} \). If \( C(t) = 1 \) and \( N(t) > 0 \), then \( R(t) \) represent the elapsed retrial time. If \( C(t) = 2 \) and \( N(t) > 0 \) then \( S(t) \) corresponding to the elapsed time of the customer being served in normal busy period. If \( C(t) = 3 \) and \( N(t) > 0 \) then \( S(t) \) corresponding to the elapsed time of the customer being served in lower rate service period.

Let \( \{t_n; n = 1,2,...\} \) be the sequence of epochs at which either a normal service or lower service period completion occurs. The sequence of random vectors \( Z_n = \{C(t_n), N(t_n)\} \) forms a Markov chain which is embedded in the retrial queueing system. It follows from Appendix A that \( \{Z_n; n \in N\} \) is ergodic if and only if \( \rho < R^*(\lambda) \), for our system to be stable, where

\[
\rho = \frac{\lambda}{\delta}(1 - S^*(\delta)).
\]
For the process \( \{N(t), \ t \geq 0\} \), we define the probabilities \( P_0(t) = P\{C(t) = 0, N(t) = 0\} \) and \( W_0(t) = P\{C(t) = 0, N(t) = 0\} \) the probability densities

\[
R_n(x,t)dx = P\{C(t) = 1, N(t) = n, x \leq R^0(t) < x + dx\}, \text{ for } t \geq 0, x \geq 0 \text{ and } n \geq 1.
\]
\[
\Pi_n(x,t)dx = P\{C(t) = 2, N(t) = n, x \leq S^0_n(t) < x + dx\}, \text{ for } t \geq 0, x \geq 0, n \geq 0.
\]
\[
W_n(x,t)dx = P\{C(t) = 4, N(t) = n, x \leq S^0_n(t) < x + dx\}, \text{ for } t \geq 0, x \geq 0, \text{ and } n \geq 0.
\]

We assume that the stability condition is fulfilled in the sequel and so that we can set

\[
P_0 = \lim_{t \to \infty} P_0(t) \text{ and } W_0 = \lim_{t \to \infty} W_0(t) \text{ limiting densities for } x > 0 \text{ and } n \geq 0
\]

\[
R_n(x) = \lim_{t \to \infty} R_n(x,t), \quad \Pi_n(x) = \lim_{t \to \infty} \Pi_n(x,t) \quad \text{and} \quad W_n(x) = \lim_{t \to \infty} W_n(x,t).
\]

### 3.1. The steady state equations

The system of governing equations of server states as follows:

\[
\lambda P_0 = \eta W_0. \tag{3.1}
\]

\[
(\lambda + \theta + \eta)W_0 = \theta Q_0 + q \int_0^\infty \Pi_0(x)\mu(x)dx + q \int_0^\infty W_0(x)\mu_\infty(x)dx + \delta \int_0^\infty \Pi_\infty(x)dx, \ n \geq 0. \tag{3.2}
\]

\[
\frac{dR_n(x)}{dx} + (\lambda + r(x)) R_n(x) = 0, \ n \geq 1. \tag{3.3}
\]

\[
\frac{d\Pi_n(x)}{dx} + (\lambda + \delta + \mu(x)) \Pi_n(x) = 0, \ n = 0. \tag{3.3}
\]

\[
\frac{d\Pi_n(x)}{dx} + (\lambda + \delta + \mu(x)) \Pi_n(x) = \lambda \Pi_{n+1}(x), \ n \geq 1. \tag{3.4}
\]

\[
\frac{dW_0(x)}{dx} + (\lambda + \theta + \eta + \mu_\infty(x)) W_0(x) = 0, \ n = 0. \tag{3.5}
\]
\[
\frac{dW_n(x)}{dx} + (\lambda + \theta + \eta + \mu_n(x))W_n(x) = \lambda W_{n-1}(x), \quad n \geq 1. \tag{3.6}
\]

The steady state boundary conditions at \( x = 0 \) are

\[
R_n(0) = p \int_0^\infty \Pi_n(x) \mu(x) dx + q \int_0^\infty \Pi_{n-1}(x) \mu(x) dx + p \int_0^\infty W_n(x) \mu_n(x) dx + q \int_0^\infty W_{n-1}(x) \mu_n(x) dx, \quad n \geq 1. \tag{3.7}
\]

\[
\Pi_0(0) = \int_0^\infty R_0(x) dx + (\theta + \eta) \int_0^\infty W_0(x) dx + \lambda P_0, \quad n = 0. \tag{3.8}
\]

\[
\Pi_0(0) = \int_0^\infty R_0(x) dx + (\theta + \eta) \int_0^\infty R_0(x) dx + (\theta + \eta) \int_0^\infty W_0(x) dx, \quad n \geq 1. \tag{3.9}
\]

\[
W_n(0) = \begin{cases} 
\lambda W_0, & n = 0 \\
0, & n \geq 1
\end{cases} \tag{3.10}
\]

The normalizing condition is

\[
P_0 + W_0 + \sum_{n=1}^\infty \int_0^\infty R_n(x) dx + \sum_{n=0}^\infty \left( \int_0^\infty \Pi_n(x) dx + \int_0^\infty W_n(x) dx \right) = 1 \tag{3.11}
\]

3.2. Computation of the steady state solution

In the following, the probability generating function technique is used here to obtain the steady state solution of the retrial queueing model. To solve the above equations, we define the generating functions for \(|z| \leq 1\), as follows:

\[
R(x,z) = \sum_{n=1}^\infty R_n(x) z^n; \quad R(0,z) = \sum_{n=1}^\infty R_n(0) z^n; \quad \Pi(x,z) = \sum_{n=0}^\infty \Pi_n(x) z^n; \quad \Pi(0,z) = \sum_{n=0}^\infty \Pi_n(0) z^n; \quad W(x,z) = \sum_{n=0}^\infty W_n(x) z^n
\]

and \(W(0,z) = \sum_{n=0}^\infty W_n(0) z^n\).
Multiplying the steady state equation and steady state boundary condition (3.2) - (3.10) by \( z^n \) and summing over \( n \), \( (n = 0,1,2,...) \) and solving the partial differential equations, it follows that

\[ \frac{\partial R(x,z)}{\partial x} + (\lambda + r(x)) R(x,z) = 0 \]  
\( (3.12) \)

\[ \frac{\partial \Pi(x,z)}{\partial x} + (\lambda(1-z) + \delta + \mu(x)) \Pi(x,z) = 0 \]  
\( (3.13) \)

\[ \frac{\partial W(x,z)}{\partial x} + (\lambda(1-z) + \eta + \mu_n(x)) W(x,z) = 0 \]  
\( (3.14) \)

\[ R(0,z) = (pz + q) \int_0^\infty \Pi(x,z)\mu(x)dx + (pz + q) \int_0^\infty W(x,z)\mu_n(x)dx - \delta \int_0^\infty \Pi(x,z)dx - (\lambda + \eta)W_0 \]  
\( (3.15) \)

\[ \Pi(0,z) = \frac{1}{z} \int_0^\infty R(x,z)r(x)dx + \lambda \int_0^\infty R(x,z)dx + (\theta + \eta) \int_0^\infty W(x,z)dx + \lambda P_0 \]  
\( (3.16) \)

\[ W(0,z) = \lambda W_0 \]  
\( (3.17) \)

Solving the partial differential equations (3.12)-(3.14), it follows that

\[ R(x,z) = R(0,z)[1 - R(x)]e^{-\lambda x} \]  
\( (3.18) \)

\[ \Pi(x,z) = \Pi(0,z)[1 - S(x)]e^{-\lambda(x)} \]  
\( (3.19) \)

\[ W(x,z) = W(0,z)[1 - S_n(x)]e^{-A(x)z} \]  
\( (3.20) \)

where \( A(z) = (\delta + \lambda(1-z)) \) and \( A_n(z) = (\theta + \eta + \lambda(1-z)) \).

Inserting equation (3.18)-(3.20) and (3.15) and make some calculation, finally we get,

\[ \Pi(0,z) = \frac{R(0,z)}{z} \left[ R'(\lambda) + z(1 - R'(\lambda)) \right] + \lambda P_0 + \lambda W_0 \]  
\( (3.21) \)
\[
V(z) = \frac{(\theta + \eta)[1 - S'_w(A_w(z))]}{(\theta + \eta) + \lambda(1 - z)} \quad \text{and} \quad S(z) = \frac{\delta[1 - S'(A(z))]}{\delta + \lambda(1 - z)}
\]

Using (3.18)-(3.20) and (3.21) in (3.15), we get

\[
R(0, z) = (pz + q) \Pi(0, z)\left( S'(A(z)) + S(z) \right) + (pz + q)W(0, z)S'_w(A_w(z)) - (\lambda + \eta)W_0 \tag{3.22}
\]

Using (3.17) and (3.21) in (3.22), we get

\[
\left( z - (pz + q)\left(R'(\lambda) + z(1 - R'(\lambda))\right)\left(S'(A(z)) + S(z)\right) \right)R(0, z) = zW_0\left(\left(S'(A(z)) + S(z)\right)(\lambda V(z) + \eta) + \lambda \left((pz + q)S'_w(A_w(z)) - 1\right) - \eta \right) \tag{3.23}
\]

From the above equation, we know that the key element for obtaining \(P(0, z)\) is to find the zeros of \(f(z) = z - (pz + q)\left(R'(\lambda) + z(1 - R'(\lambda))\right)\left(S'(A(z)) + S(z)\right) = 0\) in the range \(0 < z < 1\) for the equation \(f(z) = 0\) (from Gao et al. (2014)). From this, we give the lemma in Appendix B.

From (3.23), we get

\[
R(0, z) = \frac{zW_0\left(\left(S'(A(z)) + S(z)\right)(\lambda V(z) + \eta) + \lambda \left((pz + q)S'_w(A_w(z)) - 1\right) - \eta \right)}{\left( z - (pz + q)\left(R'(\lambda) + z(1 - R'(\lambda))\right)\left(S'(A(z)) + S(z)\right) \right)} \tag{3.24}
\]

Using the equation (3.24) in (3.21), we get

\[
\Pi(0, z) = W_0\left( \frac{z(\lambda V(z) + \eta) + \lambda \left((pz + q)S'_w(A_w(z)) - 1\right) - \eta \right)\left(R'(\lambda) + z(1 - R'(\lambda))\right)}{\left( z - (pz + q)\left(R'(\lambda) + z(1 - R'(\lambda))\right)\left(S'(A(z)) + S(z)\right) \right)} \tag{3.25}
\]

Using (3.17) and (3.24)-(3.25) in (3.18)-(3.20), then limiting PGFs \(R(x, z), \Pi(x, z)\) and \(W(x, z)\).
3.3. The steady state results

If the system is in steady state condition \( \rho < R'(\lambda), \)

The probability generating function of number of customers in the orbit when server is idle

\[
R(z) = \int_0^\infty R(x, z)dx = \frac{zW_o\left(1-R'(\lambda)\right)\left(S'(A(z))+S(z)\right)(\lambda V(z)+\eta)+\lambda\left((pz+q)S'_o(A_o(z))-1-\eta\right)}{\lambda\left\{z-(pz+q)\left[R'(\lambda)+z(1-R'(\lambda))\right]\left(S'(A(z))+S(z)\right)\right\}}
\] (3.26)

The probability generating function of number of customers in the orbit when server is regular busy

\[
\Pi(z) = \int_0^\infty \Pi(x, z)dx = \frac{W_o\left(1-S'(A(z))\right)\left(z(\lambda V(z)+\eta)+\lambda\left((pz+q)S'_o(A_o(z))-1-\eta\right)\left[R'(\lambda)+z(1-R'(\lambda))\right]\right]}{A(z)\left\{z-(pz+q)\left[R'(\lambda)+z(1-R'(\lambda))\right]\left(S'(A(z))+S(z)\right)\right\}}
\] (3.27)

The probability generating function of number of customers in the orbit when server is lower speed service

\[
W(z) = \int_0^\infty W(x, z)dx = \frac{\lambda W_s V(z)}{(\theta + \eta)}.
\] (3.28)

Using the normalizing condition, we can be determined \( P_0 \) and \( W_0 \), by setting \( z = 1 \) in (3.26)-(3.28) and applying L-Hospitals rule whenever necessary and we get \( P_0 + W_0 + R(1) + \Pi(1) + W(1) = 1. \)

The probability that server is idle in lower speed service,

\[
W_0 = \frac{R'(\lambda)-p-\lambda}{\delta}\left(1-S'(\delta)\right).
\] (3.29)

The probability that server is idle in regular service,
Corollary 3.1 If the system satisfies the steady state condition,

The probability generating function of the number of customer in the system \((K_s(z))\) is obtained by using

\[
P_0 = \frac{R'(\lambda) - p \frac{\lambda}{\delta} (1 - S'(\delta))}{\left( q + \frac{\eta}{\lambda} \right) R'(\lambda) + \frac{\lambda}{(\theta + \eta)} \left( 1 - S'_u(\theta + \eta) \right) \left( q + 1 - S'_u(\theta + \eta) \left( 1 - R'(\lambda) \right) \right) - \frac{\lambda}{\delta} \left( 1 - S'(\delta) \right)}
\]  
(3.30)

4. System performance measures

Our analysis is based on the following system characteristics of the retrial queueing system.

4.1. System state probabilities

(i) Let \(R\) be the steady state probability that the server is idle during the retrial

\[
R = R(l) = \frac{W_0 \left( 1 - R'(\lambda) \right) \left[ p + \left( 1 - S'_u(\theta + \eta) \right) \left( 1 - S'(\delta) \right) + \frac{\lambda}{\delta} \left( 1 - S'(\delta) \right) \right]}{R'(\lambda) - \rho}
\]

(ii) Let \(\Pi\) be the steady-state probability that the server is busy

\[
\Pi = \Pi(l) = \frac{W_0 \left( 1 - S'(\delta) \right) \left[ p S'_u(\theta + \eta) + \lambda \left( 1 - S'_u(\theta + \eta) \right) \left( R'(\lambda) \right) + \frac{\lambda}{(\theta + \eta)} \left( 1 - S'(\delta) \right) \right]}{R'(\lambda) - \rho}
\]

(iii) Let \(W\) be the steady state probability that the server is on lower speed service
\[ W = W(l) = \frac{\lambda W_0 (1 - S_w'(\theta + \eta))}{(\theta + \eta)}. \]

(iv) Let \( W_{wb} \) be the steady state probability that the server is on WVB,

\[ W_{wb} = W + W_0 = \frac{W_0 ((\theta + \eta) + \lambda (1 - S_w'(\theta + \eta)))}{(\theta + \eta)}. \]

(v) Let \( F_f \) be the steady state probability that server failure,

\[ F_f = \delta \times \Pi(l) = \frac{W_0 (1 - S'(\delta)) \left\{ p S_w'(\theta + \eta) + \lambda (1 - S_w'(\theta + \eta)) \right\} \left( \frac{\lambda}{(\theta + \eta)} + R'(\lambda) \right) + \eta R'(\lambda)}{(R'(\lambda) - \rho)}. \]

### 4.2. Mean system size and orbit size

(i) The expected number of customers in the orbit \( (L_q) \) is obtained by differentiating (3.32) with respect to \( z \) and evaluating at \( z = 1 \)

\[ L_q = K'_q(l) = \lim_{z \to 1} \frac{d}{dz} K_q(z). \]

(ii) The expected number of customers in the system \( (L_s) \) is obtained by differentiating (3.31) with respect to \( z \) and evaluating at \( z = 1 \)

\[ L_s = K'_s(l) = \lim_{z \to 1} \frac{d}{dz} K_s(z). \]

(iii) The average time a customer spends in the system \( (W_s) \) and the average time a customer spends in the queue \( (W_q) \) are found by using the Little’s formula

\[ W_s = \frac{L_s}{\lambda} \quad \text{and} \quad W_q = \frac{L_q}{\lambda}. \]

### 4.3. Mean busy period and Mean busy cycle

Let \( E(T_b) \) and \( E(T_c) \) be the expected length of busy period and busy cycle under the steady state conditions. The results follow directly by applying the argument of an alternating renewal process which leads to
\[ P_0 = \frac{E(T_0)}{E(T_0) + E(T_b)}, E(T_b) = 1 - \left( \frac{1}{P_0} - 1 \right) \quad \text{and} \quad E(T_e) = \frac{1}{\lambda P_0} = E(T_0) + E(T_b). \]  

(4.1)

where \( T_0 \) is the time length that the system is in empty state. Since the inter-arrival time between two customers follows exponential distribution with parameter \( \lambda \), we have that \( E(T_0) = 1/\lambda \).

Inserting (3.30) into (4.1) and use the above results, then we can get

\[
E(T_b) = \frac{1}{\lambda \alpha} \frac{\left\{ \left[ q + \frac{\delta}{\lambda} \right] R^*(\lambda) + \frac{\delta}{(\theta + \delta)} \left( 1 - S^*_w(\theta + \delta) \right) \left( q + 1 - S^*_w(\theta + \delta) \left( 1 - R^*(\lambda) \right) \right) - \frac{1}{\alpha} S^*_w(\theta + \delta) \left( 1 - S^*_w(\delta) \right) \right\}}{R^*(\lambda) - p - \frac{1}{\delta} \left( 1 - S^*_w(\delta) \right)}.
\]

(4.2)

\[
E(T_e) = \frac{1}{\delta \alpha} \frac{\left\{ \left[ q + \frac{\delta}{\lambda} \right] R^*(\lambda) + \frac{\lambda}{(\theta + \delta)} \left( 1 - S^*_w(\theta + \delta) \right) \left( q + 1 - S^*_w(\theta + \delta) \left( 1 - R^*(\lambda) \right) \right) - \frac{2}{\alpha} S^*_w(\theta + \delta) \left( 1 - S^*_w(\delta) \right) \right\}}{R^*(\lambda) - p - \frac{2}{\alpha} \left( 1 - S^*_w(\delta) \right)}.
\]

(4.3)

5. Special cases

We present three special cases of our model in following.

Case (i): No negative arrival, No feedback, No repair and No working breakdown

Let \( \alpha = \delta = c = 0 \); our model can be reduced to an M/G/1 retrial queue with working vacations. The results coincide with the result of Gao et al. (2014).

Case (ii): No negative arrival, No feedback, No repair and No working breakdown

Let \( (\alpha, \delta, \theta, p) \to (0, 0, 0, 0) \); our model can be reduced to M/G/1 retrial queue with single working vacation. This model results coincide with Arivudainambi et al. (2014).

Case (iii): No negative arrival, No feedback, No repair, No working breakdown and vacation
Let \((\alpha, \delta, \theta, p) \rightarrow (0, 0, 0, 0)\); our model can be reduced to M/G/1 retrial queue with general retrial times. The following result coincides with the result of Gomes Corral (1999).

6. Numerical examples

In this section, based on the results obtained, we present some numerical examples using MATLAB in order to illustrate the effect of various parameters in the system performance measures. Without loss of generality, we assume that the retrial times, service times, vacation times and repair times are exponential, 2-stage Erlang and 2-stage hyper-exponential distributed with the parameters \(\alpha, p\) and \(\theta\). The arbitrary values to the parameters are so chosen such that they satisfy the stability condition.

The following tables give the computed values of various characteristics of our model like, probability that the server is idle \((P_0)\), the mean orbit size \((L_q)\), probability that server is idle during retrial time \((R)\), busy \((II)\) and working breakdown \((W)\) respectively. The exponential distribution is \(f(x) = \phi e^{-\phi x}, x > 0\), Erlang distribution of order 2 is \(f(x) = \phi^2 xe^{-\phi x}, x > 0\) and hyper-exponential distribution of order 2 is \(f(x) = c\phi e^{-\phi x} + (1-c)\phi^2 e^{-\phi^2 x}, x > 0\) and \(0 < c < 1\).

In Table 1, we show that the effect of failure rate \((\alpha)\) on \(P_0\) and \(L_q\). When the system affecting time increase, the probability of no patients in the buffer increases and the number of patients waiting in the buffer decreases. That is, if the negative arrival rate increases, the server is idle \((P_0)\) increases, the coefficient of variation \((\rho)\) decreases, the mean orbit size \((L_q)\) increases and probability that the server is busy \((II)\) also increases for the values of \(\lambda = 2, a = 3, \mu_b = 4, \mu_w = 2, p = 0.5, \theta = 1, \delta = 4;\)

In Table 2 with the increase of feedback probability \((p)\), then the probability that server is idle \((P_0)\) decreases, the coefficient of variation \((\rho)\) increases, and the mean orbit size \((L_q)\) increases.
that is, as the number of patients increases for retransmission, probability of no patients in the
waiting line decreases and the number of packets in the line increases for the values of
\( \lambda = 2, \ a = 3, \ \mu_w = 4, \ \mu_w = 2, \ \theta = 1, \ p = 0.5, \ \alpha = 0.3, \ \delta = 4; \)

Table 3 shows that when the vacation rate (\( \theta \)) increases, then the probability that server is idle
\( (P_0) \) increases, the coefficient of variation (\( \rho \)) decreases, the mean orbit size \( (L_q) \) decreases and
the probability that server is busy in working vacation \( (W) \) also decreases for the values of
\( \lambda = 2, \ a = 3, \ \mu_w = 4, \ \mu_w = 2, \ p = 0.5, \ \alpha = 0.3, \ \delta = 4; \)

For the effect of the parameters \( \lambda, \ r, \ \theta, \ \eta, \ \mu, \ \mu_w \) and \( \delta \) on the system performance measures,
three dimensional graphs are illustrated in Figure 1 – Figure 4. In Figure 1, we examine the
behaviour of the mean orbit size \( (L_q) \) decreases for increasing the value of the lower service rate
\( (\mu_w) \) and regular service rate \( (\mu) \). The surface displays an upward trend as expected for increasing
the value of the of lower speed service rate \( (\mu_w) \) and repair rate \( (\eta) \) against the idle probability \( P_0 \)
in Figure 2. As expected from Figure 3, the surface displays a downward trend as expected for
increasing the value of arrival rate \( (\lambda) \) and negative arrival rate \( (\delta) \) against the mean orbit size \( L_q \)
in Figure 7. In Figure 4, we examine the behaviour of the mean orbit size \( (L_q) \) decreases for
increasing the value of vacation rate \( (\theta) \) and retrial rate \( (r) \).

From the above numerical examples, we can find the influence of parameters on the
performance measures in the system.

7. Conclusion

In this work, we have studied an M/G/1 retrial G-queue with feedback under working
vacations and working breakdowns (M/G/1/WVB). Applying the PGF approach and the
supplementary variable technique, the probability generating functions for the numbers of
customers in the system and its orbit when it is free, regular busy, on lower speed service are derived. Various system performance measures and some important special cases are discussed. The explicit expressions for the average queue length of orbit and system have been obtained. Finally, some numerical examples are presented to study the impact of the system parameters.

The novelty of this investigation is the introduction of working breakdown queueing models in presence of retrial queues with multiple working vacations. This proposed model has potential practical real life application in production to order system to enhance the performance of the production facility and to stop the production facility from becoming overloaded, in computer processing system and telephone consultation of medical service systems.

Appendix A

The embedded Markov chain \({Z_n; n \in N}\) is ergodic if and only if \(\rho < R^*(\lambda)\), where

\[
\rho = p + \frac{\lambda}{\delta} \left(1 - S^*(\delta)\right).
\]

**Proof.** To prove the sufficient condition of ergodicity, it is very convenient to use Foster’s criterion (Pakes, 1969), which states that the chain \({Z_n; n \in N}\) is an irreducible and aperiodic Markov chain is ergodic if there exists a non-negative function \(f(j), j \in N\) and \(\epsilon > 0\), such that mean drift \(\psi_j = E[f(z_{n+1}) - f(z_n) / z_n = j]\) is finite for all \(j \in N\) and \(\psi_j \leq -\epsilon\) for all \(j \in N\), except perhaps for a finite number \(j\)’s. In our case, we consider the function \(f(j) = j\). then we have

\[
\psi_j = \begin{cases} 
\rho - 1, & \text{if } j = 0, \\
\rho - R^*(\lambda), & \text{if } j = 1, 2, ...
\end{cases}
\]

Clearly the inequality \(\rho < R^*(\lambda)\) is sufficient condition for Ergodicity.
To prove the necessary condition, as noted in Sennott et al. (1983), if the Markov chain \( \{Z_n; n \geq 1\} \) satisfies Kaplan’s condition, namely, \( \varphi_j < \infty \) for all \( j \geq 0 \) and there exits \( j_0 \in \mathbb{N} \) such that \( \varphi_j \geq 0 \) for \( j \geq j_0 \). Notice that, in our case, Kaplan’s condition is satisfied because there is a \( k \) such that \( m_{ij} = 0 \) for \( j < i - k \) and \( i > 0 \), where \( M = (m_{ij}) \) is the one step transition matrix of \( \{Z_n; n \in \mathbb{N}\} \). Then \( \rho \geq R'(\lambda) \) implies the non-Ergodicity of the Markov chain.

**Appendix B**

**Lemma 3.1.** If \( \rho < R'(\lambda) \), the equation \( z - (pz + q)(R^*(\lambda) + z(1 - R^*(\lambda))\left( S'(A(z)) + S(z) \right) \) has no roots in the range \( 0 < z < 1 \) and has the minimal nonnegative root \( z = 1 \).

**Proof.** We only need to prove that

\[
u(z) = (pz + q)(R^*(\lambda) + z(1 - R^*(\lambda))\left( S'(A(z)) + S(z) \right)
\]

is a probability generating function of the number of customers that arrive in the system. Denote by \( U \) the time period from the epoch a service completion occurs, leaving the orbit non-empty, to the next service completion epoch, by \( N_U \) the number of primary customers that arrive during \( U \) and define

\[
u_j(t) = P(t < U \leq t + dt, N(U) = j).
\]

Then,

\[
u_j(t) = e^{-\lambda t} \alpha(t) * a_j(t) + (1 - \delta_{j,0}) \lambda e^{-\lambda t} (1 - R(t)) * a_{j-1}(t), j = 0, 1, 2...
\]

where \( * \) means convolution, \( \alpha(t) \) is the p.d.f. of inter-retrial times, \( b(t) \) is the p.d.f. of normal service times and \( a_j(t)dt = e^{-\lambda t} (\lambda t)^j/j! b(t) \). Denote by \( N_U(z) \) the probability generating function of \( N_U \), we have that
\[ N_U(z) = \sum_{j=0}^{\infty} \int_{0}^{\infty} u_j(t) dt \]
\[ = \sum_{j=0}^{\infty} \int_{0}^{\infty} \left( e^{-\lambda t} \alpha(t) + (1 - \delta_{j,0}) \lambda e^{-\lambda t} (1 - R(t)) + a_j(t) \right) dt \]
\[ = u(z), \]

which proves the expected result that \( u(z) = (pz + q)\left( R'(\lambda) + z(1 - R'(\lambda)) \right) S'(A(z)) + S(z) \) is exactly a probability generating function. From assumption \( \rho < R'(\lambda) \), we have \( E[N_u] = \frac{d}{dz} u(z) \big|_{z=1} = 1 - \left( R'(\lambda) - \rho \right) < 1 \).

and the convex function \( u(z) \) is a monotonically increasing function of \( z \) for \( 0 \leq z \leq 1 \), and \( u(0) = P(N_U = 0) < 1 \), \( u(1) = 1 \). So we can easily prove the expected result of Lemma 3.1.

Then for \( \rho < R'(\lambda) \), \( z - (pz + q)\left( R'(\lambda) + z(1 - R'(\lambda)) \right) S'(A(z)) + S(z) \) never vanishes in the range \( 0 < z < 1 \).

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Figure 1. $L_q$ versus $\mu$ and $\mu_w$

Figure 2. $P_0$ versus $\mu_w$ and $\eta$
Figure 3. $L_q$ versus $\lambda$ and $\alpha$

Figure 4. $L_q$ versus $r$ and $\theta$
Table 1. The effect of failure rate ($\alpha$) on $P_0$ and $L_q$.

<table>
<thead>
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<th>failure rate ($\alpha$)</th>
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<th>Hyp- Exp</th>
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Table 2. The effect of feedback probability ($p$) on $P_0$ and $L_q$.

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<th>Hyp- Exp</th>
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Table 3. The effect of vacation rate ($\theta$) on $P_0$ and $L_q$.

<table>
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