### On $w^*$-Connected Spaces

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For Proof Read only
ON ω*-CONNECTED SPACES

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Abstract

In this paper, we introduce the notion of ω-separated sets and ω*-connected spaces. We obtain several properties of the notion analogous to those of connectedness. We show that the continuous image of the ω*-connected space is connected.

Keywords:
ω-open set, ω-separated, ω-connected, ω-component.

1 Introduction

Let (X, τ) be a topological space with no separation properties assumed. For a subset H of a topological space (X, τ), Cl(H) and Int(H) denote the closure and the interior of H in (X, τ), respectively.

Definition 1.1. (Kuratowski, 1933) Let H be a subset of a topological space (X, τ). A point p in X is called a condensation point of H if, for each open set U containing p, U ∩ H is uncountable.
**Definition 1.2.** (Hdeib, 1982) A subset \( H \) of a topological space \((X, \tau)\) is called \( \omega \)-closed if it contains all its condensation points.

The complement of an \( \omega \)-closed set is called \( \omega \)-open.

It is well known that a subset \( W \) of a space \((X, \tau)\) is \( \omega \)-open if and only if for each \( x \in W \), there exists \( U \in \tau \) such that \( x \in U \), and \( U - W \) is countable (Hdeib, 1989). The family of all \( \omega \)-open sets that is denoted by \( \tau_\omega \) is a topology on \( X \), which is finer than \( \tau \). The interior and closure operators in \((X, \tau_\omega)\) are denoted by \( \text{Int}_\omega \) and \( \text{Cl}_\omega \), respectively. Many topological concepts and results related to the \( \omega \)-closed and \( \omega \)-open sets appeared in Al Ghour and Zareer (2016), Al-Omari and Noorani (2007a, 2007b), Noiri, Al-Omari and Noorani (2009a, 2009b), Zorlutuna (2013) and in the references therein. In this paper, we introduce the notion of \( \omega \)-separated sets and \( \omega^* \)-connected spaces. We obtain several properties of the notion analogous to these of connectedness. We show that the continuous image of the \( \omega^* \)-connected space is connected. Furthermore, we present a connected space that is not \( \omega^* \)-connected.

2. \( \omega \)-SEPARATED SETS

**Definition 2.1.** Nonempty subsets \( A \) and \( B \) of a topological space \((X, \tau)\). The pair \((A, B)\) are called \( \omega \)-separated if \( \text{Cl}(A) \cap B = A \cap \text{Cl}_\omega(B) = \phi \).

Clearly, every separated set is \( \omega \)-separated, the converse need not be true in general as the following example shown that.

**Example 2.2.** Let \( X = \{a, b, c, d\} \) and \( \tau = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}. \) If \( A = \{b\} \) and \( B = \{a\} \), then the pair \((A, B)\) form an \( \omega \)-separated set, but it is not separated. It is clear that the pair \((B, A)\) are not \( \omega \)-separated set.
**Proposition 2.3.** Let $A$ be a nonempty open set in a space $X$ and $B$ be a nonempty $\omega$-open set in a space $X$ such that $A \cap B = \emptyset$, then the pair $(A, B)$ are $\omega$-separated.

Proof. Let $A \cap B = \emptyset$. Then, $A \subseteq X - B$ and $\text{Cl}_\omega(A) \subseteq \text{Cl}_\omega(X - B) = X - B$, which implies that $\text{Cl}_\omega(A) \cap B = \emptyset$. Again, $B \subseteq X - A$ implies that $\text{Cl}(B) \subseteq \text{Cl}(X - A) = X - A$ and so $\text{Cl}(B) \cap A = \emptyset$. Therefore, the pair $(A, B)$ are $\omega$-separated.

**Corollary 2.4.** Let $(X, \tau)$ be topological space. If $A, B \in \tau$ are nonempty open sets such that $A \cap B = \emptyset$, then $A$ and $B$ are $\omega$-separated.

**Proposition 2.5.** Let the pair $(A, B)$ be two $\omega$-separated subsets in a topological space $(X, \tau)$. If $C, D \in X$ are nonempty subsets such that $C \subseteq A$ and $D \subseteq B$, then the pair $(C, D)$ are also $\omega$-separated.

Proof. Suppose that the pair $(A, B)$ are $\omega$-separated and $\text{Cl}(A) \cap B = \emptyset = A \cap \text{Cl}_\omega(B)$. Now, $C \cap \text{Cl}_\omega(D) \subseteq A \cap \text{Cl}_\omega(B) = \emptyset$, so $C \cap \text{Cl}_\omega(D) = \emptyset$. Similarly, we can prove that $\text{Cl}(C) \cap D = \emptyset$. Hence, the pair $(C, D)$ are $\omega$-separated.

**Theorem 2.6.** Let $(X, \tau)$ be a topological space. If $A$ and $B$ are $\omega$-separated such that $A \cup B$ is closed set, then one set is closed, and the other is $\omega$-closed.

Proof. Let the pair $(A, B)$ be $\omega$-separated sets and $A \cup B$ is closed. Then, $A \cap \text{Cl}_\omega(B) = \emptyset = \text{Cl}(A) \cap B$. For every $A \cup B$ is closed, $A \cup B = \text{Cl}(A) \cup \text{Cl}(B)$. Now, $\text{Cl}(A) = \text{Cl}(A) \cap [\text{Cl}(A) \cup \text{Cl}(B)] = \text{Cl}(A) \cap [A \cup B] = [\text{Cl}(A) \cap A] \cup [\text{Cl}(A) \cap B] = A \cup \emptyset = A$, hence $A$ is closed. Additionally, $B \subseteq A \cup B$ then we have:

$$\text{Cl}_\omega(B) \subseteq \text{Cl}_\omega(A \cup B) \subseteq \text{Cl}(A \cup B) = A \cup B$$

so $\text{Cl}_\omega(B) = \text{Cl}_\omega(B) \cap [A \cup B] = [\text{Cl}_\omega(B) \cap A] \cup [\text{Cl}(B) \cap B] = \emptyset \cup B = B$. 


Hence, $B$ is $\omega$-closed.

**Theorem 2.7.** Let $(X, \tau)$ be a topological space. If the pair $(A, B)$ are $\omega$-separated sets of $X$ and $A \cup B \in \tau$, then $A$ and $B$ are $\omega$-open and open, respectively.

Proof. Let the pair $(A, B)$ be $\omega$-separated in $X$; then, $B = [A \cup B] \cap [X \setminus Cl(A)]$. Since $A \cup B \in \tau$ and $Cl(A)$ is closed in $X$, then $B$ is open. Thus, $A = [A \cup B] \cap [X \setminus Cl_{\omega}(B)]$ since the pair $(A, B)$ are $\omega$-separated in $X$. Additionally, $A \cup B \in \tau \subseteq \tau_{\omega}$ and $Cl_{\omega}(B)$ is $\omega$-closed in $X$, and then $A$ is $\omega$-open.

**Lemma 2.8.** (Al-Omari and Noorani (2007a)) Let $(X, \tau)$ be topological space if $Y$ is an open subspace of a space $X$ and $B \subseteq Y \subseteq X$. Then, $Cl_{\omega}(Y) = Cl_{\omega}(B) \cap Y$.

**Lemma 2.9.** Let $(X, \tau)$ be topological space and $Y$ is an open subspace of a space $X$ such that $A, B \subseteq Y \subseteq X$. The following statements are equivalent:

1. The pair $(A, B)$ are $\omega$-separated in $Y$;
2. The pair $(A, B)$ are $\omega$-separated in $X$.

Proof. This is obvious from Lemma 2.8, $Cl_{\omega}(A) \cap B = \phi = A \cap Cl_{\omega}(B)$ if and only if $Cl_{\omega}(A) \cap B = \phi = A \cap Cl(B)$.

**3 $\omega^*$-CONNECTED SPACES**

In this section, we discuss some properties of $\omega^*$-connected space, which is stronger than connected space.

**Definition 3.1.** A subset $A$ of a topological space $(X, \tau)$ is called $\omega^*$-connected if $A$ is not the union of any pair of $\omega$-separated sets in $(X, \tau)$.
Clearly, every $\omega^*$-connected space is connected, the converse need not be true in general as the following example shown that.

**Example 3.2.** Let $\mathbb{R}$ be the set of real numbers and $X = \mathbb{Q}$ be the set of all rational numbers. Let $(X, \tau_{l|\mathbb{Q}})$ be the relative topology with left-ray topology $(\mathbb{R}, \tau_l)$. Then, $(X, \tau_{l|\mathbb{Q}})$ is a connected space, but it is not $\omega^*$-connected.

**Example 3.3.** Let $\mathbb{R}$ be the set of real numbers and $\mathbb{Q}$ be the set of all rational numbers $(\mathbb{R} - \mathbb{Q} = I$ be the set of all irrational numbers) with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$. Then, $(\mathbb{R}, \tau)$ is connected space, but it is not $\omega^*$-connected since the pair $(I, \mathbb{Q})$ are $\omega$-separated sets.

**Theorem 3.4.** A topological space $(X, \tau)$ is $\omega^*$-connected if and only if $X$ cannot be written as the disjoint union of a nonempty $\omega$-open set and a nonempty open set.

Proof. Suppose that $X$ is not a union of nonempty disjoint $\omega$-open and open sets $A$ and $B$ that is $X = A \cup B$. Thus, $Cl(A) \cap B = \phi = A \cap Cl_\omega(B)$ since $A$ and $B$ are disjoint. Hence the pair $(A, B)$ are $\omega$-separated sets in $X$. So, $X$ is not $\omega^*$-connected. This is a contradiction.

Conversely, Suppose that $X$ is not $\omega^*$-connected. There exist a pair $(A, B)$ of $\omega$-separated sets and $X = A \cup B$. By Theorem 2.7, $A, B \in X$ are $\omega$-open and open, respectively. Then, $X$ can be written as the disjoint union of a nonempty $\omega$-open set and a nonempty open set. This is a contradiction.

**Theorem 3.5.** A subspace $Y$ of a topological space $(X, \tau)$ is $\omega^*$-connected if and only if there does not exist any pair of $\omega$-separation for $Y$. 

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Proof. First, let us assume that the subspace $Y$ is $\omega^*$-connected. Thus, we will have to prove that $Y$ does not admit any $\omega$-separation. Suppose that $Y$ has a pair of $\omega$-separation. Hence, there exist nonempty subsets $A$ and $B$ of $X$ such that $Y = A \cup B$, $A \cap Cl(B) = \phi = Cl_\omega(A) \cap B$. Now,

$$Cl_\omega(Y) = Cl_\omega(A) \cap Y = Cl_\omega(A) \cap [A \cup B] = [Cl_\omega(A) \cap A] \cup [Cl_\omega(A) \cap B] = A$$

This implies that $A$ is an $\omega$-closed subset of $(Y, \tau_Y)$. Similarly,

$$Cl_\omega(Y) = Cl(Y) \cap Y = Cl(Y) \cap [A \cup B] = [Cl(Y) \cap A] \cup [Cl(Y) \cap B] = B$$

This implies that $B$ is a closed subset of $(Y, \tau_Y)$. Additionally, $Y - B = Y \cap [X - B] = [A \cup B] \cap [X - B] = [A \cap (X - B)] \cup [(X - B) \cap B] = A \cap (X - B) = A$

(since $A \cap B = \phi$). This means that the complement of $B$ with respect to $Y$ is $A$. Hence, $B$ is $\omega$-open such that $B \neq \phi$ and $B \neq Y$ (since $Y = A \cup B$ and $A \neq \phi$), and $B$ is both $\omega$-open and closed in $Y$. This implies that the subspace $(Y, \tau_Y)$ is not an $\omega^*$-connected space. This is a contradiction, and we arrived at this contradiction by assuming that $Y$ has a pair of $\omega$-separation. Hence, there does not exist any $\omega$-separation for $Y$.

Conversely, now assume that there is not any pair of $\omega$-separation for $Y$. Now suppose that the subspace $(Y, \tau_Y)$ is not an $\omega$-connected space. Then, there exist nonempty $\omega$-closed set $A$ and closed set $B$ in $(Y, \tau_Y)$ such that $Y = A \cup B$ and $A \cap B = \phi$.

Now,

$$[Cl_\omega(A) \cap B = Cl_\omega(A) \cap [Y \cap B] = [Cl_\omega(A) \cap Y] \cap B = Cl_\omega(A) \cap B = A \cap B = \phi$$

(since $A$ is $\omega$-closed in $Y$, which implies $Cl_\omega(Y) = A$). Similarly, $A \cap Cl(B) = \phi$. This means that $Y$ has a pair of $\omega$-separation, and this is a contradiction. We arrived at this contradiction by assuming that the subspace $(Y, \tau_Y)$ is not an $\omega^*$-connected space. Hence,
the assumption is wrong. This means that \((Y, \tau_Y)\) is \(\omega^*\)-connected.

**Theorem 3.6.** Let \((X, \tau)\) not be an \(\omega^*\)-connected topological space and \(\phi \neq A \subseteq X\), \(A\) is an open set and \(\omega\)-closed set in \(X\). Suppose, \(\phi \neq Y\) is \(\omega^*\)-connected subspace of \(X\). Then, either \(Y \subseteq A\) or \(Y \subseteq X - A\).

Proof. \(X = A \cup B\), where \(B = X - A\) implies that:
\[
Y = X \cap Y = [A \cup B] \cap Y = (A \cap Y) \cup (B \cap Y).
\]

Also \([A \cap Y] \cap CL[B \cap Y] \subseteq A \cap CL(B) = A \cap B = \phi\), which implies that \([A \cap Y] \cap CL[B \cap Y] = \phi\).

Similarly, \(CL_\omega[A \cap Y] \cap [B \cap Y] \subseteq CL_\omega(A) \cap B = A \cap B = \phi\).

Then \(CL_\omega[A \cap Y] \cap [B \cap Y] = \phi\). It is given that \(Y\) is an \(\omega^*\)-connected subspace of \(X\). So, it not implies that \(A \cap Y \neq \phi\) and \(B \cap Y \neq \phi\). \(Y\) cannot admit any pair of \(\omega\)-separation, since \(Y\) is an \(\omega^*\)-connected subspace of \(X\). Thus, \(A \cap Y = \phi\) or \(B \cap Y = \phi\) and so \(Y \subseteq A\) or \(Y \subseteq X - A\).

**Theorem 3.7.** Let \((X, \tau)\) be a topological space and the pair \((H, G)\) are \(\omega\)-separated sets of \(X\). If \(A\) is an \(\omega^*\)-connected set of \(X\) with \(A \subseteq H \cup G\), then either \(A \subseteq H\) or \(A \subseteq G\).

Proof. Let \(A \subseteq H \cup G\). Since \(A = [A \cap H] \cup [A \cap G]\), then \([A \cap H] \cap CL_\omega[A \cap G] \subseteq H \cap CL_\omega(G) = \phi\). Again, we have \([A \cap G] \cap CL[A \cap H] \subseteq G \cap CL(H) = \phi\). Thus, the pair \((A \cap H, A \cap G)\) are \(\omega\)-separated sets. If \(A \cap H\) and \(A \cap G\) are nonempty. So, \(A\) is not \(\omega^*\)-connected. This is a contradiction. Thus, either \(A \cap H = \phi\) or \(A \cap G = \phi\). Hence \(A \subseteq H\) or \(A \subseteq G\).

**Theorem 3.8.** If \(A\) is an \(\omega^*\)-connected set of a topological space \((X, \tau)\) and \(A \subseteq B \subseteq CL_\omega(A)\), then \(B\) is \(\omega^*\)-connected.

Proof. Suppose that \(B\) is not \(\omega^*\)-connected. There exist a pair \((H, G)\) of \(\omega\)-separated sets
such that \( B = H \cup G \). So, \( H \) and \( G \) are nonempty and \( G \cap \text{Cl}(H) = \phi = H \cap \text{Cl}_\omega(G) \).

Then either \( A \subseteq H \) or \( A \subseteq G \), by Theorem 3.7. If \( A \subseteq G \), then \( \text{Cl}_\omega(A) \subseteq \text{Cl}_\omega(G) \) and \( H \cap \text{Cl}_\omega(A) = \phi \). Thus \( H \subseteq B \subseteq \text{Cl}_\omega(A) \) and \( H = \text{Cl}_\omega(A) \cap H = \phi \). Hence \( H \) is an empty set.

This is a contradiction, since \( H \) is nonempty. Now, if \( A \subseteq H \). Again, we have \( G \) is empty.

This is a contradiction. Then, \( B \) is \( \omega^* \)-connected.

**Corollary 3.9.** If \( A \) is an \( \omega^* \)-connected set of a topological space \((X, \tau)\), then \( \text{Cl}_\omega(A) \) is \( \omega^* \)-connected.

**Theorem 3.10.** Let \((X, \tau)\) be a topological space. If \( \{N_i; i \in I\} \) be a nonempty family of \( \omega^* \)-connected sets of \( X \) with \( \cap_{i \in I} N_i \neq \phi \), then \( \cup_{i \in I} N_i \) is \( \omega^* \)-connected.

Proof. Suppose that \( \cup_{i \in I} N_i \) is not \( \omega^* \)-connected. Then, \( \cup_{i \in I} N_i = H \cup G \), where the pair \((H, G)\) are \( \omega \)-separated sets in \( X \). Since \( \cap_{i \in I} N_i \neq \phi \), we have \( x \in \cap_{i \in I} N_i \). Since \( x \in \cup_{i \in I} N_i \), either \( x \in H \) or \( x \in G \). If \( x \in H \). Since \( x \in N_i \) for each \( i \in I \), then \( N_i \) and \( H \) intersect for each \( i \in I \). By Theorem 3.7, \( N_i \subseteq H \) or \( N_i \subseteq G \). Since \( H \) and \( G \) are disjoint, \( N_i \subseteq H \) for all \( i \in I \) and hence \( \cup_{i \in I} N_i \subseteq H \). Then \( G \) is empty. This is a contradiction.

Suppose \( x \in G \). In a similar way, we have that \( H \) is empty. This is a contradiction. Thus, \( \cup_{i \in I} N_i \) is \( \omega^* \)-connected.

**Theorem 3.11.** Every continuous image of an \( \omega^* \)-connected space is a connected space.

Proof. Let \( f : X \to Y \) be a continuous function and \( X \) be an \( \omega^* \)-connected space. Suppose that \( f(X) \) is not a connected subset of \( Y \). Thus, there exists nonempty separated sets \( A \) and \( B \) with \( f(X) = A \cup B \). Since \( f \) is continuous and

\[
A \cap \text{Cl}(B) = \phi = \text{Cl}(A) \cap B,
\]

\[
\text{Cl}(f^{-1}(A)) \cap f^{-1}(B) \subseteq f^{-1}(\text{Cl}(A)) \cap f^{-1}(B) = f^{-1}[\text{Cl}(A) \cap B] = \phi,
\]

\[
f^{-1}(A) \cap \text{Cl}_\omega(f^{-1}(B)) \subseteq f^{-1}(A) \cap \text{Cl}(f^{-1}(B)) \subseteq f^{-1}(A) \cap f^{-1}\text{Cl}(B)) = f^{-1}[A \cap
\]

$\text{Cl}(B) = \phi$. Since $A$ and $B$ are nonempty, $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty. Hence, $f^{-1}(A)$ and $f^{-1}(B)$ are a pair of $\omega$-separated and $X = f^{-1}(A) \cup f^{-1}(B)$. This is a contradiction since $X$ is $\omega^*$-connected. Therefore, $f(X)$ is connected.

**Question:** Is a continuous image of an $\omega^*$-connected space is an $\omega^*$-connected?

**Theorem 3.12.** If every distinct points of a subset $H$ of a space $X$ are elements of some $\omega^*$-connected subset of $H$, then $H$ is an $\omega^*$-connected of $X$.

Proof. Suppose $H$ is not $\omega^*$-connected. Then, there exist nonempty sets $A$, $B \in X$ and $\text{Cl}(A) \cap B = A \cap \text{Cl}_\omega(B)$ and $H = A \cup B$. Since $A$ and $B$ are nonempty, there exists $a \in A$ and $b \in B$. By hypothesis, $a$ and $b$ must be elements of an $\omega^*$-connected subset $C$ of $H$. Since $C \subseteq A \cup B$, by Theorem 3.7, either $C \subseteq A$ or $C \subseteq B$. Consequently, either both $a$ and $b$ are in $A$ or in $B$. Let $a$, $b \in A$. Hence, $A \cap B \neq \phi$, which is a contradiction to the fact that $A$ and $B$ are disjoint. Therefore, $H$ must be $\omega^*$-connected.

**Theorem 3.13.** If $A$ is an $\omega^*$-connected subset of an $\omega^*$-connected topological space $(X, \tau)$ such that $X - A$ is the union of a pair $(B, C)$ of $\omega$-separated sets, then $A \cup C$ are $\omega^*$-connected.

Proof. Suppose $A \cup B$ is not $\omega^*$-connected. Then, there exist a pair $(G, H)$ of nonempty $\omega$-separated sets such that $A \cup B = G \cup H$. Thus, $A \subseteq A \cup B = G \cup H$, since $A$ is $\omega^*$-connected and, by Theorem 3.7, either $A \subseteq G$ or $A \subseteq H$. If $A \subseteq G$. Since $A \cup B = G \cup H$, $A \subseteq G$, then $A \cup B \subseteq G \cup H$ and hence $G \cup H \subseteq G \cup B$. Hence, $H \subseteq B$. Since a pair $(B, C)$ are $\omega$-separated, also a pair $(H, C)$ are $\omega$-separated. Thus, $H$ is $\omega$-separated from a pair$(G, C)$. Now,

$\text{Cl}(H) \cap [G \cup C] = [\text{Cl}(H) \cap G] \cup [\text{Cl}(H) \cap C] = \phi$, and

$H \cap \text{Cl}_\omega[G \cup C] = H \cap [\text{Cl}_\omega(G) \cup \text{Cl}_\omega(C)] = [H \cap \text{Cl}_\omega(G)] \cup [H \cap \text{Cl}_\omega(C)] = \phi$. 


Therefore, $H$ is $\omega$-separated from $G \cup C$. Since $X - A = B \cup C$, $X = A \cup [B \cup C] = [A \cup B] \cup C = [G \cup H] \cup C$, and since $A \cup B = G \cup H$, $X = [G \cup C] \cup H$. Thus, $X$ is the union of a pair of nonempty $\omega$-separated sets $(G \cup C, H)$, which is a contradiction. Similar contradiction if $A \subseteq H$. Then, $A \cup B$ is $\omega^*$-connected. Also, we can prove that $A \cup C$ is $\omega^*$-connected.

**Theorem 3.14.** If $A$ and $B$ are $\omega^*$-connected sets of a topological space $(X, \tau)$ and none of them is $\omega$-separated, we have $A \cup B$ is $\omega^*$-connected.

Proof. Let $A$ and $B$ be $\omega^*$-connected in a space $X$. If $A \cup B$ is not $\omega^*$-connected. Then, there exist a pair of nonempty disjoint $\omega$-separated sets $(G, H)$ and $A \cup B = G \cup H$. By Theorem 3.7 and since $A$ and $B$ are $\omega^*$-connected, either $A \subseteq G$ and $B \subseteq H$ or $B \subseteq G$ and $A \subseteq H$. Now, if $A \subseteq G$ and $B \subseteq H$, then $A \cap H = B \cap G = \phi$.

Therefore, $[A \cup B] \cap G = [A \cap G] \cup [B \cap G] = [A \cap G] \cup \phi = A \cap G = A$.

Additionally, $[A \cup B] \cap H = [A \cap H] \cup [B \cap H] = \phi \cup [B \cap H] = B \cap H = B$.

Similarly, if $B \subseteq G$ and $A \subseteq H$, then $[A \cup B] \cap G = B$ and $[A \cup B] \cap H = A$. Now, $[(A \cup B) \cap H] \cap Cl_\omega[(A \cup B) \cap G] \subseteq (A \cup B) \cap H \cap Cl_\omega(A \cup B) \cap Cl_\omega(G) = (A \cup B) \cap H \cap Cl_\omega(G) = \phi$ and $Cl[(A \cup B) \cap H] \cap [(A \cup B) \cap G] \subseteq Cl(A \cup B) \cap Cl(H) \cap \phi = \phi$.

Therefore, the pair $(A \cup B) \cap H, (A \cup B) \cap G$ are $\omega$-separated sets. Thus, by Proposition 2.5, we have that the pair $(A, B)$ are $\omega$-separated, which is a contradiction. Hence, $A \cup B$ is $\omega^*$-connected.

**Example 3.15.** Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{b\}, \{b, c\}, \{a, b, d\}\}$. If $A = \{c\}$ and $B = \{d\}$, the $A$ and $B$ are $\omega^*$-connected. But $A \cup B = \{c, d\}$ is not $\omega^*$-connected since the pairs $(A, B)$ and $(B, A)$ are $\omega$-separated.
Example 3.16. Let \( X = \{a, b, c, d\} \), \( \tau = \{ X, \phi, \{b\}, \{b, c\}, \{a, b, d\}\} \). If \( A = \{b\} \) then \( A \) is \( \omega^* \)-connected. But \( Cl(A) = X \) is not \( \omega^* \)-connected since the pair \((H, G)\) are \( \omega \)-separated where \( H = \{a, c, d\} \) and \( G = \{b\} \).

Definition 3.17. Let \((X, \tau)\) be topological space and \( x \in X \). The union of all \( \omega^* \)-connected subsets of \( X \) containing \( x \) is called the \( \omega \)-component of \( X \) containing \( x \).

Lemma 3.18. Each \( \omega \)-component of a topological space \((X, \tau)\) is a maximal \( \omega^* \)-connected set of \( X \).

Lemma 3.19. The set of all distinct \( \omega \)-components of a topological space \((X, \tau)\) forms a partition of \( X \).

Proof. Let \( A \) and \( B \) be distinct \( \omega \)-components of \( X \). If \( A \) and \( B \) intersect. Then, by Theorem 3.10, \( A \cup B \) is \( \omega^* \)-connected in \( X \). Since \( A \subseteq A \cup B \), then \( A \) is not maximal. Hence, \( A \) and \( B \) are disjoint.

Lemma 3.20. Each \( \omega \)-component of a topological space \((X, \tau)\) is an \( \omega \)-closed in \( X \).

Proof. Let \( A \) be an \( \omega \)-component of \( X \). By Corollary 3.9, \( Cl_\omega(A) \) is \( \omega^* \)-connected and \( A = Cl_\omega(A) \). Thus, \( A \) is an \( \omega \)-closed in \( X \).

Example 3.21. Let \( X = \{a, b, c, d\} \), \( \tau = \{ X, \phi, \{b\}, \{b, c\}, \{a, b, d\}\} \). If \( A = \{b\} \) then the \( \omega \)-component of \( A \) is \( A \) which is not closed.

Theorem 3.22. Each \( \omega^* \)-connected subset of a space \( X \) which open and \( \omega \)-closed is \( \omega \)-component of \( X \).

Proof. Let \( A \) be an \( \omega^* \)-connected of \( X \) which open and \( \omega \)-closed. For \( x \in A \). If \( C \) is the \( \omega \)-
component containing $x$, then $A \subseteq C$ (since $A$ is an $\omega^*$-connected subset of $X$ containing $x$). Let $A \subseteq C$. Then, $C \neq \emptyset$ and $C \cap (X - A) \neq \emptyset$. Thus, $X - A$ is closed and $\omega$-open and $[A \cap C] \cap [(X - A) \cap C] = \emptyset$ (since $A$ is open and $\omega$-closed). Additionally, $[A \cap C] \cup [(X - A) \cap C] = [A \cup (X - A)] \cap C = C$. Also, $A$ and $X - A$ are nonempty disjoint open and $\omega$-open sets, respectively, and $A \cap \text{Cl}(X - A) = \emptyset = \text{Cl}_\omega(A) \cap (X - A)$. Then we have that $(A \cap C) \cap \text{Cl}[(X - A) \cap C] = \emptyset = \text{Cl}_\omega(A \cap C) \cap [(X - A) \cap C]$. Hence the pair $[A \cap C]$ and $[(X - A) \cap C]$ are $\omega$-separated sets. This is a contradiction, then $A = C$. Hence, $A$ is an $\omega$-component of $X$.

4. Conclusions

In the present work, we have continued to study the properties of connected spaces. We introduced $\omega$-separated sets and $\omega^*$-connected. Moreover, we have also established several results and presented their fundamental properties with the help of some examples.

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References


