New transform formulae for differential transformation method with applications to the nonlinear plane autonomous systems
Original Article

New transform formulae for differential transformation method

with applications to the nonlinear plane autonomous systems

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Abstract

This work presents a new derivation technique for new differential transform formulae of a product of composite functions. The new formulae are applied to nonlinear plane autonomous systems to demonstrate their efficiency and reliability. The approximate series solutions estimated by the differential transform method (DTM) and the multistep differential transform method (MsDTM) are then compared with the flow direction of the vector fields defined by the original system and an analytical solution calculated by the phase-plane method. We found that the MsDTM results are in better agreement with the analytical solution than the DTM ones. Moreover, the MsDTM can be applied to systems whose analytical solutions are unobtainable. The approximate solutions by the MsDTM have same direction to the flow of the vector field of the system. It follows that proposed new formulae are reliable and efficient.
Keywords: differential transform method, nonlinear plane autonomous systems, multistep differential transform method, phase-plane method

1. Introduction

Autonomous systems are systems of first-order DEs of the form

\[
\begin{align*}
\frac{dx_1}{dt} &= g_1(x_1,\ldots,x_n) \\
\frac{dx_2}{dt} &= g_2(x_1,\ldots,x_n) \\
&\vdots \\
\frac{dx_n}{dt} &= g_n(x_1,\ldots,x_n)
\end{align*}
\]

such that the independent variable does not explicitly appear on the right hand side of each DE. In the case \( n = 2 \), the system is called a plane autonomous system and \( V(x_1, x_2) = (g_1(x_1, x_2), g_2(x_1, x_2)) \) is a vector field in the plane that indicates the movement direction. If the parameter \( t \) is interpreted as time, then \( X(t) = (x(t), y(t)) \) indicates the position of the particle in plane at time \( t \) and a solution of the system is interpreted as a path of this particle starting from \( X(0,0) = (x(0), y(0)) \) (Zill & Wright, 2014).

The differential transformation method (DTM) is an alternative procedure for obtaining an approximate Taylor series solution of differential equations. The main advantage of this method is that it can be applied directly to nonlinear differential equations without the requiring linearization and discretization. The concept of the differential transform method was introduced by Zhou (Zhou, 1986), who solved linear and nonlinear problems in electrical circuits and many other problems related to differential equations (see also Damirchi & Shamami, 2016; Mahgoub & Alshikh, 2017;
Methi, 2016; Mirzaee, 2011; Moon, Bhosale, Gajbhiye, & Lonare, 2014; Patil & Khambayat, 2014).

Although the DTM series solution gives a good approximation for some problems, in some cases, the series solution diverges in a wider domain. Due to this reason the multistep differential transform method (MsDTM) is used. The MsDTM is based on the DTM, but compared with other methods, it does not need small parameters, auxiliary functions and parameters, discretization. In this technique, the solution domain is divided in subdomains (see Ebenezer, Freihet, Khan & Khan, 2016; Ertürk, Odibat & Momami, 2012; Odibat, Bertelle, Aziz-Alaoui & Duchamp, 2010; Rashidi, Chamkha, & Keimanesh, 2011; Zurigat & Ababneh, 2015). In particular, we are interested in the technique introduced by Chang (Change & Chang, 2008), for calculating the DTM of nonlinear functions.

In this paper, a derivation technique of new differential transform formulae for the product of composite functions is presented. The computation consists of three steps. The first step is finding the differential transformation of the product of two composite functions in the general form as shown in Eq. (3.1). The next step is finding the differential transformation for the higher order derivative of a power function as shown in Lemma 3.1. The last step is the derivation of the new differential transformation shown Formulae 1-8, calculated by using the general formulae of higher order derivatives of composite functions studied in (Weisstein) combined with Lemma 3.1. Then, the new differential transform formulae obtained are used to transform the nonlinear plane autonomous systems for finding the DTM and the MsDTM approximate solutions of the problem. By comparing graphically the results, we obtain that the approximate series solutions calculated by the DTM and the MsDTM have the same
direction with the vector fields flow and they are also similar to analytical solution obtained by phase-plane method.

Here is the structure of the paper. In section 2, the one-dimensional differential transformation method is described. In section 3, the analysis of the method and new formulae calculation are proposed. In section 4, the new differential transform formulae proposed are applied to three examples of nonlinear plane autonomous systems to show the reliability and efficiency of the method. The conclusion is given at the end of the paper in section 5.

2. One-Dimensional Differential transform method

The basic definitions and fundamental operations of the differential transform are introduced as follows.

**Definition 2.1** The one-dimensional differential transform of the function $x(t)$ is defined as

$$X(k) = \frac{1}{k!} \left[ \frac{d^k x(t)}{dt^k} \right]_{t=t_0}, \quad k \geq 0. \tag{2.1}$$

In Eq. (2.1), $x(t)$ is called the original function and $X(k)$ is called the transformed function.

**Definition 2.2** The inverse one-dimensional differential transform of $X(k)$ is defined as

$$x(t) = \sum_{k=0}^{\infty} X(k)(t-t_0)^k, \tag{2.2}$$

that is,

$$x(t) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} \left[ \frac{d^k x(t)}{dt^k} \right]_{t=t_0}. \tag{2.3}$$
Equation (2.3) implies that the concept of differential transformation method is derived from Taylor series expansion. Actually, in concrete applications, the function $x(t)$ is expressed by a truncated series and Eq. (2.2) becomes

$$x(t) = \sum_{k=0}^{N} X(k)(t-t_0)^k.$$  

(2.4)

The fundamental operations of one-dimensional the DTM are shown in Table 1.

The multistep differential transformation method (MsDTM) is advantageous for applications in Physics. For instance, due to small time steps the MsDTM has a powerful accuracy especially for initial value problem (IVP).

Let $[0, T]$ be the interval over which we want to find the solution of the IVP. In actual applications of the DTM, the approximate solution of the IVP can be expressed by the finite series

$$x(t) = \sum_{k=0}^{N} X(k)(t)^k, \quad t \in [0, T].$$  

(2.5)

Let us assume that the interval $[0, T]$ is divided into $n$ subintervals $[t_{i-1}, t_i], \ i = 1, \ldots, m$ of the equal step size $h = T / m$ by using the nodes $t_i = ih$. The main ideas of the MsDTM can be found in (Odibat, Bertelle, Aziz-Alaoui, & Duchamp, 2010). In fact, the MsDTM gives us the solution in the form,

$$x(t) = \begin{cases} 
    x_0(t), & t \in [0, t_1] \\
    x_1(t), & t \in [t_1, t_2] \\
    \vdots \\
    x_m(t), & t \in [t_m, t_{m-1}], 
\end{cases}$$  

(2.6)

where $x_i(t) = \sum_{k=0}^{N} X_i(k)(t-t_i)^k$ and the initial condition $x_i^{(k)}(t_{i-1}) = x_{i-1}^{(k)}(t_{i-1})$.

3. Analysis of Method
In this section, we will introduce our derivation technique of the new differential transform formulae for the product of composite functions as derived in Formulae 1-8.

To obtain these new formulae, the derivation is shown as in the following steps.

**Step 1** The differential transformation for the product of two composite functions represented by $f(y(t))g(y(t))$ which are the original functions. By definition 2.1 of the DTM combined with Leibniz formula, we obtain

\[
\frac{1}{k!} \left[ \frac{d^k}{dt^k} f(y(t))g(y(t)) \right] = \frac{1}{k!} \left[ \sum_{r=0}^{k} \frac{k!}{(k-r)!} \frac{d^r}{dt^r} f(y(t)) \frac{d^{k-r}}{dt^{k-r}} g(y(t)) \right]_{t=t_0}
\]

\[
= \sum_{r=0}^{k} F(r)G(k-r),
\]

where

\[
F(r) = \frac{1}{r!} \left[ \frac{d^r}{dt^r} f(y(t)) \right]_{t=t_0},
\]

\[
G(k-r) = \frac{1}{(k-r)!} \left[ \frac{d^{k-r}}{dt^{k-r}} g(y(t)) \right]_{t=t_0}.
\]

**Step 2** This step is finding the differential transformation for the higher order derivative of the power function that will be used in Step 3.

**Lemma 3.1** If $k, r, m \in \mathbb{N}^+ \cup \{0\}$ and let $w = r - m = 0, \ldots, r$ where $r = 0, \ldots, k$, $m = 0, \ldots, r$,

then

\[
\left[ \frac{1}{k!} \frac{d^k}{dt^k} y(t)^w \right]_{t=t_0} = 1, \quad k = 0,
\]

\[
\left[ \frac{1}{k!} \frac{d^k}{dt^k} y(t)^w \right]_{t=t_0} = \sum_{k_{w-1}=0}^{k} \sum_{k_{w-2}=0}^{k-1} \cdots \sum_{k_1=0}^{k} Y(k_1)Y(k_2-k_1)\cdots Y(k-k_{w-1}), \quad k > 0. \quad (3.2)
\]

Proof. Assume that $k, r, m \in \mathbb{N}^+ \cup \{0\}$ and let $w = r - m = 0, \ldots, r$ where $r = 0, \ldots, k$, $m = 0, \ldots, r$. 

Case $k = 0$; we have $r = 0$ and $m = 0$, then

$$\left[ \frac{1}{0!} \frac{d^0 y(t)^0}{dt^0} \right]_{t=t_0} = 1.$$ 

Case $k > 0$; we will prove by mathematical induction. Let $P(w)$ be Eq. (3.2).

First, we will show that the statement holds for $w = 0$, that is

$$P(0) = \left[ \frac{1}{k!} \frac{d^k y(t)^0}{dt^k} \right]_{t=t_0} = 0.$$ 

Next, we assume that the statement is true for $w = r - 1$, that is

$$P(r-1) = \left[ \frac{1}{k!} \frac{d^k y(t)^{r-1}}{dt^k} \right]_{t=t_0} = \sum_{k_{r-1}=0}^{k} \sum_{k_{r}=0}^{k-k_{r-1}} \cdots \sum_{k_{1}=0}^{k} Y(k_{1})Y(k_{2} - k_{1}) \cdots Y(k - k_{r-1}).$$

We will show that the statement is also true for $w = r$. This can be seen as follows

$$P(r) = \left[ \frac{1}{k!} \frac{d^k y(t)^r}{dt^k} \right]_{t=t_0} = \left[ \frac{1}{k!} \frac{d^k (y(t)^r y(t))}{dt^k} \right]_{t=t_0}$$

$$= \left[ \frac{1}{k!} \sum_{k_{r-1}=0}^{k} \frac{k!}{(k-k_{r-1})! k_{r-1}!} \frac{d^{k-k_{r-1}}}{dt^{k-k_{r-1}}} y(t)^{r-1} \frac{1}{(k-k_{r-1})!} \frac{d^{k-k_{r-1}}}{dt^{k-k_{r-1}}} y(t) \right]_{t=t_0}$$

$$= \sum_{k_{r-1}=0}^{k} \sum_{k_{r}=0}^{k-k_{r-1}} \cdots \sum_{k_{1}=0}^{k} Y(k_{1})Y(k_{2} - k_{1}) \cdots Y(k - k_{r-1}).$$

Therefore, the statement holds for $w = r$, the proof is completed.

Step 3 The functions $f(y(t))$ and $g(y(t))$ in Step 1 are be considered as the original functions in the Formulae 1-8. To obtain these new differential transform formulae, the general formulae of higher order derivatives of some composite functions are used together with Lemma 3.1 in the following calculation.
**Formula 1** If \( f(y(t)) = e^{y(t)} \) is the original function, then

\[
F(k) = \frac{1}{k!} \left[ \frac{d^k}{dt^k} e^{y(t)} \right] = \frac{1}{k!} \left[ e^{y(t)} \sum_{r=0}^{k} \frac{1}{r!} \sum_{m=0}^{r} \frac{(-1)^m r!}{(r-m)!m!} y^m(t) \frac{d^k}{dt^k} y(y(t))^{r-m} \right]
\]

\[
= e^{y(0)} \sum_{r=0}^{k} \frac{1}{r!} \sum_{m=0}^{r} \frac{(-1)^m y^m(t_0)}{(r-m)!m!} \left[ \frac{1}{k!} \frac{d^k}{dt^k} (y(t))^{r-m} \right]
\]

\[
= e^{y(0)} \sum_{r=0}^{k} \frac{1}{r!} \sum_{m=0}^{r} \frac{(-1)^m Y^m(0)}{(r-m)!m!} \left[ \frac{1}{k!} \frac{d^k}{dt^k} (y(t))^{r-m} \right],
\]

where \( Y(0) = y(t_0) \), and we have used Lemma 3.1 to transform \( \left[ \frac{1}{k!} \frac{d^k}{dt^k} (y(t))^{r-m} \right] \).

**Formula 2** If \( f(y(t)) = \ln(y(t)) \), \( y(t) > 0 \) is the original function, then

\[
F(k) = \frac{1}{k!} \left[ \frac{d^k}{dt^k} \ln(y(t)) \right] = \frac{1}{k!} \left[ \delta_k \ln(y(t))^{(k)} + \sum_{r=1}^{k} \frac{(-1)^{r-i}}{r!} \left( \frac{k}{r} \right) \frac{1}{k!} \frac{d^k}{dt^k} (y(t))^{(r)} \right]
\]

\[
= \frac{1}{k!} \delta_k \ln(y(0))^{(k)} + \sum_{r=1}^{k} \frac{(-1)^{r-i}}{r!} \left( \frac{k}{r} \right) \left[ \frac{1}{k!} \frac{d^k}{dt^k} (y(t))^{(r)} \right],
\]

where \( \binom{k}{r} = \frac{k!}{(k-r)!r!} \) are the binomial coefficients, \( \delta_k = \begin{cases} 1, & k = 0 \\ 0, & k = 1, 2, 3, \ldots \end{cases} \)

\[
\left[ \frac{1}{k!} \frac{d^k}{dt^k} (y(t))^{(r)} \right]_{t=t_0} = \sum_{k_1=0}^{k} \sum_{k_2=0}^{k-k_1} \ldots \sum_{k_r=0}^{k-k_2-k_{r-1}} Y(k_1)Y(k_2-k_1)\ldots Y(k-r)...
\]

**Formula 3** If \( f(y(t)) = \sin(y(t)) \) is the original function, then

\[
F(k) = \frac{1}{k!} \left[ \frac{d^k}{dt^k} \sin(y(t)) \right] = \frac{1}{k!} \left[ \sum_{r=0}^{k} \frac{1}{r!} \frac{d^r}{dt^r} \sin(t) \right] \left[ \sum_{m=0}^{r} \frac{(-1)^m r!}{(r-m)!m!} y^m(t) \frac{d^k}{dt^k} y(y(t))^{r-m} \right]
\]

\[
= \sum_{r=0}^{k} \frac{d^r}{dt^r} \sin(t) \sum_{m=0}^{r} \frac{(-1)^m y^m(t_0)}{(r-m)!m!} \left[ \frac{1}{k!} \frac{d^k}{dt^k} (y(t))^{r-m} \right]_{t=t_0}
\]
\[
\sum_{r=0}^{k} \frac{d^r}{dt^r} \sin(t) \sum_{m=0}^{(r-m)!m!} \frac{(1)^m Y^m(0)}{(r-m)!m!} \left[ \frac{1}{k!} \frac{d^k}{dt^k} (Y(t))^{r-m} \right]_{t=t_0}
\]

where \( Y(0) = y(t_0) \), and we have used Lemma 3.1 to transform \( \left[ \frac{1}{k!} \frac{d^k}{dt^k} (Y(t))^{r-m} \right]_{t=t_0} \).

**Formula 4** If \( f(y(t)) = \cos(y(t)) \) is the original function, then

\[
F(k) = \frac{1}{k!} \left[ \frac{d^k}{dt^k} \cos(Y(t)) \right]_{t=t_0} = \frac{1}{k!} \left[ \sum_{r=0}^{k} \frac{1}{r!} \frac{d^r}{dt^r} \cos(Y(t)) \sum_{m=0}^{(r-m)!m!} \frac{(1)^m Y^m(0)}{(r-m)!m!} \left[ \frac{1}{k!} \frac{d^k}{dt^k} (Y(t))^{r-m} \right]_{t=t_0} \right]
\]

\[
= \sum_{r=0}^{k} \frac{d^r}{dt^r} \cos(t) \sum_{m=0}^{(r-m)!m!} \frac{(1)^m Y^m(t_0)}{(r-m)!m!} \left[ \frac{1}{k!} \frac{d^k}{dt^k} (Y(t))^{r-m} \right]_{t=t_0}
\]

where \( Y(0) = y(t_0) \), and we have used Lemma 3.1 to transform \( \left[ \frac{1}{k!} \frac{d^k}{dt^k} (Y(t))^{r-m} \right]_{t=t_0} \).

**Formula 5** If \( f(y(t)) = \sin(y(t)) \) is the original function, then

\[
F(k) = \frac{1}{k!} \left[ \frac{d^k}{dt^k} \sin(Y(t)) \right]_{t=t_0} = \frac{1}{k!} \left[ \sum_{r=0}^{k} \frac{1}{r!} \frac{d^r}{dt^r} \sin(Y(t)) \sum_{m=0}^{(r-m)!m!} \frac{(1)^m Y^m(0)}{(r-m)!m!} \left[ \frac{1}{k!} \frac{d^k}{dt^k} (Y(t))^{r-m} \right]_{t=t_0} \right]
\]

\[
= \sum_{r=0}^{k} \frac{d^r}{dt^r} \sin(t) \sum_{m=0}^{(r-m)!m!} \frac{(1)^m Y^m(t_0)}{(r-m)!m!} \left[ \frac{1}{k!} \frac{d^k}{dt^k} (Y(t))^{r-m} \right]_{t=t_0}
\]

where \( Y(0) = y(t_0) \), and we have used Lemma 3.1 to transform \( \left[ \frac{1}{k!} \frac{d^k}{dt^k} (Y(t))^{r-m} \right]_{t=t_0} \).

**Formula 6** If \( f(y(t)) = \cosh(y(t)) \) is the original function, then

\[
F(k) = \frac{1}{k!} \left[ \frac{d^k}{dt^k} \cosh(Y(t)) \right]_{t=t_0} = \frac{1}{k!} \left[ \sum_{r=0}^{k} \frac{1}{r!} \frac{d^r}{dt^r} \cosh(Y(t)) \sum_{m=0}^{(r-m)!m!} \frac{(1)^m Y^m(0)}{(r-m)!m!} \left[ \frac{1}{k!} \frac{d^k}{dt^k} (Y(t))^{r-m} \right]_{t=t_0} \right]
\]
\[
\begin{align*}
&= \sum_{r=0}^{k} \frac{d^r}{dt^r} \cosh(t) \left[ \sum_{m=0}^{r} \frac{(-1)^m y^{(m)}(t_0)}{(r-m)!m!} \left( \frac{1}{k!} \frac{d^k}{dt^k} (y(t))^{r-m} \right) \right]_t =_t \\
&= \sum_{r=0}^{k} \frac{d^r}{dt^r} \cosh(t) \left[ \sum_{m=0}^{r} \frac{(-1)^m y^{(m)}(0)}{(r-m)!m!} \left( \frac{1}{k!} \frac{d^k}{dt^k} (y(t))^{r-m} \right) \right]_t \end{align*}
\]

where \( Y(0) = y(t_0) \), and we have used Lemma 3.1 to transform \( \left[ \frac{1}{k!} \frac{d^k}{dt^k} (y(t))^{r-m} \right]_t \).

**Formula 7** If \( f(y(t)) = \sqrt{y(t)} \) is the original function, then

\[
F(k) = \frac{1}{k!} \left[ \frac{d^k}{dt^k} \sqrt{y(t)} \right]_t = \frac{1}{k!} \left[ \frac{\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)\Gamma(\frac{1}{2})} \sum_{r=0}^{k} \frac{(-1)^r}{(r-\frac{1}{2})!r!} \left( \frac{k}{r} \right) (y(t))^{r-\frac{1}{2}} \frac{d^r}{dt^r} (y(t))^r \right]_t
\]

where \( \left( \frac{k}{r} \right) = \frac{k!}{(k-r)!r!} \) are the binomial coefficients,

\( \Gamma(1+z) = z \Gamma(z), z \in \mathbb{C}, \Gamma(\frac{1}{2}) = \sqrt{\pi}, \Gamma(n) = (n-1)!, n \in \mathbb{Z}^+, \) and

\[
\left[ \frac{1}{k!} \frac{d^k}{dt^k} (y(t))^r \right]_t = \sum_{k_{k-1}=0}^{k} \sum_{k_{k-2}=0}^{k-1} \ldots \sum_{k_{1}=0}^{k_{k-1}} Y(k_1)Y(k_2-k_1)...Y(k_{k-1}).
\]

**Formula 8** If \( f(y(t)) = \frac{1}{y(t)} \) is the original function, then

\[
F(k) = \frac{1}{k!} \left[ \frac{d^k}{dt^k} \frac{1}{y(t)} \right]_t = \frac{1}{k!} \left[ (k+1) \sum_{r=0}^{k} \frac{(-1)^r}{(r+1)!r!} \left( \frac{k}{r} \right) (y(t))^{-\frac{r+1}{2}} \frac{d^r}{dt^r} (y(t))^r \right]_t
\]

\[
= (k+1) \sum_{r=0}^{k} \frac{(-1)^r}{(r+1)!r!} \left( \frac{k}{r} \right) (y(t_0))^{-\frac{r+1}{2}} \left[ \frac{d^r}{dt^r} (y(t))^r \right]_t
\]

where \( \left( \frac{k}{r} \right) = \frac{k!}{(k-r)!r!} \) are the binomial coefficients, and

\[
\left[ \frac{1}{k!} \frac{d^k}{dt^k} (y(t))^r \right]_t = \sum_{k_{k-1}=0}^{k} \sum_{k_{k-2}=0}^{k-1} \ldots \sum_{k_{1}=0}^{k_{k-1}} Y(k_1)Y(k_2-k_1)...Y(k_{k-1}).
\]
The transformed functions are shown in Table 2.

4. Application

In this section, we extended the application of the DTM to nonlinear plane autonomous systems. To demonstrate the formulae introduced in the previous section, three examples are studied here. The accuracy of the method is assessed by graphical and data value comparisons.

Example 4.1 Consider the following system of nonlinear plane autonomous

\[
\begin{align*}
    x' & = e^y, \\
    y' & = e^x, \text{ for } t \in [0,1.25],
\end{align*}
\]

subject to the initial conditions \(x(0) = 0, y(0) = 0\).

Applying the DTM of Eqs. (4.1)-(4.2) and using the initial conditions \(x(0) = 0, y(0) = 0\), it follows

\[
\begin{align*}
    X(k+1) & = \frac{1}{k+1} \left( e^{y(0)} \sum_{r=0}^{k} \sum_{m=0}^{r} \frac{(-1)^m Y^m(0)}{(r-m)!m!} \left[ \frac{1}{k!} \frac{d^k}{dt^k} (y(t))^{r-m} \right] \right) \\
    Y(k+1) & = \frac{1}{k+1} \left( e^{x(0)} \sum_{r=0}^{k} \sum_{m=0}^{r} \frac{(-1)^m X^m(0)}{(r-m)!m!} \left[ \frac{1}{k!} \frac{d^k}{dt^k} (x(t))^{r-m} \right] \right).
\end{align*}
\]

\(X(0) = 0, Y(0) = 0\).

By substituting \(k = 0, ..., 11\) we obtain the coefficients of the series solution as follows

\[
\begin{align*}
    X(1) & = Y(1) = 1, X(2) = Y(2) = \frac{1}{2}, X(3) = Y(3) = \frac{1}{3}, X(4) = Y(4) = \frac{1}{4}, \\
    X(5) & = Y(5) = \frac{1}{5}, X(6) = Y(6) = \frac{1}{6}, X(7) = Y(7) = \frac{1}{7}, X(8) = Y(8) = \frac{1}{8}, \\
    X(9) & = Y(9) = \frac{1}{9}, X(10) = Y(10) = \frac{1}{10}, X(11) = Y(11) = \frac{1}{11}, X(12) = Y(12) = \frac{1}{12}.
\end{align*}
\]
Hence, the series solution reads

\[ y(t) = x(t) = t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4} + \frac{t^5}{5} + \frac{t^6}{6} + \frac{t^7}{7} + \frac{t^8}{8} + \frac{t^9}{9} + \frac{t^{10}}{10} + \frac{t^{11}}{11} + \frac{t^{12}}{12}, \quad t \in [0, 1.25]. \]

On the other hand, by applying the MsDTM to Eqs. (4.1), (4.2) with same initial conditions, it follows

\[
X_i(k+1) = \frac{1}{k+1}\left( e^{x_i(0)} \sum_{r=0}^{k} \sum_{m=0}^{r} \frac{(-1)^mY_i^{(m)}(0)}{(r-m)!m!} \left[ \frac{1}{k!} \frac{d^k}{dt^k} \left( y(t) \right)^{r-m} \right]_{t=a_i} \right),
\]

\[
Y_i(k+1) = \frac{1}{k+1}\left( e^{x_i(0)} \sum_{r=0}^{k} \sum_{m=0}^{r} \frac{(-1)^mY_i^{(m)}(0)}{(r-m)!m!} \left[ \frac{1}{k!} \frac{d^k}{dt^k} \left( x(t) \right)^{r-m} \right]_{t=a_i} \right),
\]

\[ X_0(0) = 0, X_i(0) = x_{-i}(t_i), \quad Y_0(0) = 0, Y_i(0) = y_{-i}(t_i), \quad i = 1, 2, 3, 4, 5. \]

Thus, we obtain the series solution

\[
y(t) = x(t) = \begin{cases} 
    t + 0.5t^2 + 0.3333t^3 + 0.25t^4 + 0.2t^5 + 0.16667t^6 + 0.14286t^7 \\
    + 0.125t^8 + 0.1111t^9 + 0.1t^{10} + 0.09091t^{11} + 0.0833t^{12}, \quad t \in [0, 0.25] \\
    0.286682 + 1.33333(t-0.25) + 0.888889(t-0.25)^2 + 0.790124(t-0.25)^3 \\
    + 0.790124(t-0.25)^4 + 0.842798(t-0.25)^5 + 0.936443(t-0.25)^6 \\
    + 1.07022(t-0.25)^7 + 1.24859(t-0.25)^8 + 1.47981(t-0.25)^9 \\
    + 1.77577(t-0.25)^{10} + 2.15245(t-0.25)^{11} + 2.63078(t-0.25)^{12}, \quad t \in [0.25, 0.5] \\
    0.693147 + 2(t-0.5) + 2(t-0.5)^2 + 2.66667(t-0.5)^3 + 4(t-0.5)^4 + 6.4(t-0.5)^5 \\
    + 10.6667(t-0.5)^6 + 18.2857(t-0.5)^7 + 32(t-0.5)^8 + 56.8889(t-0.5)^9 \\
    + 102.4(t-0.5)^{10} + 186.182(t-0.5)^{11} + 341.333(t-0.5)^{12}, \quad t \in [0.5, 0.75] \\
    1.38628 + 3.99993(t-0.75) + 7.99972(t-0.75)^2 + 21.3322(t-0.75)^3 \\
    + 63.9955(t-0.75)^4 + 204.782(t-0.75)^5 + 682.595(t-0.75)^6 + 2340.28(t-0.75)^7 \\
    8190.85(t-0.75)^8 + 29122.5(t-0.75)^9 + 104839(t-0.75)^10 + 381227(t-0.75)^{11} \\
    + 1.39781 \times 10^6(t-0.75)^{12}, \quad t \in [0.75, 1] \\
    4.05773 + 57.843(t-1) + 1672.91(t-1)^2 + 64510.45(t-1)^3 + 2.79861 \times 10^6(t-1)^4 \\
    + 1.56645 \times 10^7(t-1)^5 + 8.05403 \times 10^7(t-1)^6 + 4.19282 \times 10^8(t-1)^7 \\
    + 2.20478 \times 10^8(t-1)^8 + 1.16903 \times 10^9(t-1)^9, \quad t \in [1, 1.25]. 
\end{cases}
\]
This problem with the initial conditions $x(0) = 0, y(0) = 0$ can be solved analytically by phase-plane method and obtain the analytical solution $y(x) = x$. As seen in Figure 1, the approximate series solutions calculated by the DTM and the MsDTM are the same as the analytical solution and they have the same direction with the flow of the vector fields. Moreover, the DTM and the MsDTM give the data results similar to the analytical ones as shown in Table 3.

However, if we consider the problem with the another initial conditions $x(0) = -2, y(0) = 1$, the analytical solution obtained is $y(x) = \ln \left( e^x + e - e^{-2} \right)$. The data values of the approximate solutions of the DTM and the MsDTM are compared with analytical solution as shown in Table 4. We can see that the MsDTM result is much more similar to the analytical result than the DTM one.

The following two examples show that the proposed new transformed functions of the product of composite functions can be applied effectively to the nonlinear plane autonomous system when the analytical solutions are unavailable.

**Example 4.2** Let us consider the following system of nonlinear plane autonomous

\[ x' = x^2 e^x \]  \hspace{1cm} (4.3)

\[ y' = y e^x - y, \text{ for } t \in [0, 0.2], \]  \hspace{1cm} (4.4)

subject to the initial conditions $x(0) = 1, y(0) = 1$.

Applying the DTM to Eqs. (4.3)-(4.4) and with the initial conditions $x(0) = 1, y(0) = 1$, it follows

\[ X(k+1) = \frac{1}{k+1} \sum_{r=0}^{k} F(r) \sum_{l=0}^{k-r} X(l) X(k-r-l) \]

\[ Y(k+1) = \frac{1}{k+1} \left( \sum_{r=0}^{k} Y(k-r) G(r) - Y(k) \right), \]
where

\[ F(r) = e^{X(0)} \sum_{l=0}^{k} \sum_{m=0}^{l} \frac{(-1)^m Y^m(0)}{(l-m)!m!} \left[ \frac{1}{r^l} \frac{d^r}{dt^r} \left( y(t) \right)^{l-m} \right]_{t=0}^{0}, \]

\[ G(r) = e^{X(0)} \sum_{l=0}^{k} \sum_{m=0}^{l} \frac{(-1)^m X^m(0)}{(l-m)!m!} \left[ \frac{1}{r^l} \frac{d^r}{dt^r} \left( x(t) \right)^{l-m} \right]_{t=0}^{0}. \]

Hence, we obtain the series solution by the DTM

\[ x(t) = 1 + 2.71828 t + 9.72444 t^2 + 38.8048 t^3 + 164.329 t^4 + 722.872 t^5 + 3265.98 t^6 + 15052.5 t^7 + 7045.9 t^8 + 333873 t^9 + 1.59826 \times 10^6 t^{10} + 7.71576 \times 10^6 t^{11} + 3.75161 \times 10^7 t^{12}, \quad t \in [0, 0.2] \]

\[ y(t) = 1 + 1.71828 t + 5.17077 t^2 + 19.3526 t^3 + 80.1435 t^4 + 351.093 t^5 + 1593.47 t^6 + 7409.3 t^7 + 35062.4 t^8 + 168146 t^9 + 814830 t^{10} + 3.98192 \times 10^7 t^{11} + 1.95937 \times 10^8 t^{12}, \quad t \in [0, 0.2]. \]

On the other hand, by applying the MsDTM to Eqs. (4.3) and (4.4), we obtain

\[ X_i(k+1) = \frac{1}{k+1} \sum_{r=0}^{k} F_i(r) \sum_{l=0}^{k-r} X_i(l) X_i(k-r-l) \]

\[ Y_i(k+1) = \frac{1}{k+1} \left( \sum_{r=0}^{k} Y_i(k-r) G_i(r) - Y_i(k) \right), \]

\[ X_0(0) = 1, \quad X_i(0) = x_{i-1}(t_i), \quad Y_0(0) = 1, \quad Y_i(0) = y_{i-1}(t_i), \quad i = 1, 2, 3, 4 \]

where

\[ F_i(r) = e^{X(i)(0)} \sum_{l=0}^{k} \sum_{m=0}^{l} \frac{(-1)^m Y^m(0)}{(l-m)!m!} \left[ \frac{1}{r^l} \frac{d^r}{dt^r} \left( y(t) \right)^{l-m} \right]_{t=0}^{0}, \]

\[ G_i(r) = e^{X(i)(0)} \sum_{l=0}^{k} \sum_{m=0}^{l} \frac{(-1)^m X^m(0)}{(l-m)!m!} \left[ \frac{1}{r^l} \frac{d^r}{dt^r} \left( x(t) \right)^{l-m} \right]_{t=0}^{0}. \]

It results the following approximate series solution
As shown in Figure 2, the approximate series solution obtained by the MsDTM and the DTM are compared graphically with the flow direction of the vector fields. We can see that the MsDTM result is in better agreement with vector field than the DTM one.
Example 4.3 Let us consider the following system of nonlinear plane autonomous

\[ x' = 2x + \sin y \quad (4.5) \]

\[ y' = x(y^2 + 1), \text{ for } t \in [0, 0.4], \quad (4.6) \]

subject to the initial condition \( x(0) = 1, \ y(0) = 1. \)

Applying the DTM of Eqs. (4.5) and (4.6), we obtain

\[
X(k+1) = \frac{1}{k+1} \left( 2X(k) + \sum_{r=0}^{k} \frac{d^r}{dt^r} \sin(t) \right) \left[ \sum_{m=0}^{r} \frac{(-1)^m Y^m(0)}{(r-m)!m!} \left[ \frac{1}{k!} \frac{d^k}{dt^k} (y(t))^{r-m} \right] \right] \\
Y(k+1) = \frac{1}{k+1} \left( X(Y(k) + \sum_{r=0}^{k} \sum_{l=0}^{r} Y(l) Y(r-l) X(k-r) \right),
\]

and the initial condition becomes \( X(0) = 1, \ Y(0) = 1. \)

Then, we obtain the series solution

\[
x(t) = 1 + 2.84147 t + 3.38177 t^2 + 2.56549 t^3 + 0.498022 t^4 - 3.62605 t^5 - 13.1978 t^6 \\
- 36.6427 t^7 - 93.1066 t^8 - 224.685 t^9 - 520.598 t^{10} - 1160.66 t^{11} - 2481.54 t^{12}, \ t \in [0, 0.4]
\]

\[
y(t) = 1 + 2 t + 4.84147 t^2 + 10.6041 t^3 + 24.528 t^4 + 57.5466 t^5 + 134.91 t^6 \\
+ 315.164 t^7 + 733.76 t^8 + 1702.66 t^9 + 3938.03 t^{10} + 9079.66 t^{11} + 20873.9 t^{12}, \ t \in [0, 0.4].
\]

On the other hand, by applying the MsDTM to Eqs. (4.5) and (4.6), it follows

\[
X_i(k+1) = \frac{1}{k+1} \left( 2X_i(k) + \sum_{r=0}^{k} \frac{d^r}{dt^r} \sin(t) \right) \left[ \sum_{m=0}^{r} \frac{(-1)^m Y^m(0)}{(r-m)!m!} \left[ \frac{1}{k!} \frac{d^k}{dt^k} (y(t))^{r-m} \right] \right] \\
Y_i(k+1) = \frac{1}{k+1} \left( X_i(k) + \sum_{r=0}^{k} \sum_{l=0}^{r} Y_i(l) Y(r-l) X_i(k-r) \right),
\]

\( X_0(0) = 1, \ X_i(0) = x_{i-1}(t_i), \ Y_0(0) = 1, \ Y_i(0) = y_{i-1}(t_i), \ i = 1, 2, 3, 4, 5. \)

Hence, we obtain the series solution
$x(t) = \begin{cases} 1 + 2.84147t + 3.38177t^2 + 2.56549t^3 + 0.498022t^4 - 3.62605t^5 - 13.1978t^6 \\ -36.6427t^7 - 93.1066t^8 - 224.685t^9 - 520.598t^{10} - 1160.66t^{11} - 2481.54t^{12}, \ t \in [0, 0.08] \\ 1.25028 + 3.43174(t-0.08) + 3.98652(t-0.08)^2 + 2.28135(t-0.08)^3 \\ -3.27511(t-0.08)^4 - 19.3406(t-0.08)^5 - 67.1434(t-0.08)^6 \\ -20.5262(t-0.08)^7 - 584.413(t-0.08)^8 - 1563.52(t-0.08)^9 \\ -3898.43(t-0.08)^10 - 8808.52(t-0.08)^11 - 16669.2(t-0.08)^12, \ t \in [0.08, 0.16] \\ 1.55128 + 4.10089(t-0.16) + 4.24803(t-0.16)^2 - 1.14073(t-0.16)^3 \\ -23.7465(t-0.16)^4 - 108.548(t-0.16)^5 - 404.828(t-0.16)^6 \\ -1336.03(t-0.16)^7 - 3807.58(t-0.16)^8 - 8159.94(t-0.16)^9 \\ -5.19089 \times 10^6(t-0.16)^{10} + 3.98717 \times 10^6(t-0.16)^{11} - 6.53363 \times 10^{11}(t-0.16)^{12}, \ t \in [0.16, 0.24] \\ 2.27311 + 420097(t-0.32) - 9.88383(t-0.32)^2 - 38.0005(t-0.32)^3 + 1179.65(t-0.32)^4 \\ + 24086.5(t-0.32)^5 + 281349(t-0.32)^6 + 2.244486 \times 10^6(t-0.32)^7 \\ + 8.20097 \times 10^6(t-0.32)^8 - 1.15647 \times 10^9(t-0.32)^9 - 3.29081 \times 10^9(t-0.32)^{10} \\ -5.19089 \times 10^{10}(t-0.32)^{11} - 6.53363 \times 10^{11}(t-0.32)^{12}, \ t \in [0.32, 0.4]. \end{cases}$

$y(t) = \begin{cases} 1 + 2t + 4.84147t^2 + 10.6041t^3 + 24.528t^4 + 57.5466t^5 + 134.91t^6 + 315.164t^7 \\ + 733.76t^8 + 1702.66t^9 + 3938.03t^{10} + 9079.66t^{11} + 20873.9t^{12}, \ t \in [0, 0.08] \\ 1.19765 + 3.04364(t-0.08) + 8.7346(t-0.08)^2 + 24.1547(t-0.08)^3 \\ + 69.2557(t-0.08)^4 + 199.329(t-0.08)^5 + 571.305(t-0.08)^6 \\ + 1629.29(t-0.08)^7 + 4624.49(t-0.08)^8 + 13067.8(t-0.08)^9 \\ + 36779.9(t-0.08)^10 + 103170(t-0.08)^11 + 288632(t-0.08)^12, \ t \in [0.08, 0.16] \\ 1.5131 + 5.10287(t-0.16) + 18.7225(t-0.16)^2 + 68.529(t-0.16)^3 \\ + 254.775(t-0.16)^4 + 942.871(t-0.16)^5 + 3463.75(t-0.16)^6 + 12638.4(t-0.16)^7 \\ + 45855(t-0.16)^8 + 165680(t-0.16)^9 + 597192(t-0.16)^10 \\ + 2.15185 \times 10^6(t-0.16)^{11} + 7.76857 \times 10^6(t-0.16)^{12}, \ t \in [0.16, 0.24] \end{cases}$
Similarly to the previous examples, the MsDTM result is in better agreement with the flow of the vector field than the DTM one as seen in Figure 3.

5. Conclusions

The MsDTM combined with our new formulae have been successfully applied to solving nonlinear plane autonomous systems. Three different examples are solved and the series solutions of the DTM and the MsDTM are obtained. These are compared with the analytical solutions calculated by using phase-plane method in the first example and compared to the vector fields flow directions in the second and the third examples. The results show that the MsDTM are more similar to the analytical solution and to the vector field flow direction than the DTM ones. Therefore, this method based on our new transformed functions are reliable and efficient mathematical tools for solving nonlinear plane autonomous systems.

Acknowledgments

This paper has been supported by scholarships for graduate student from the Faculty of Science, KMITL.
References


Huazhong university press. Wuhan, China.


Figure 1. The MsDTM, the DTM and numerical solution compared with vector fields flow directions.

Figure 2. The MsDTM and the DTM compared with vector fields flow directions.

Figure 3. The MsDTM and the DTM compared with vector fields flow directions.
### Table 1. The fundamental operations of one-dimensional the DTM.

<table>
<thead>
<tr>
<th>Original function ( x(t) )</th>
<th>Transformed function ( X(k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x(t) \pm y(t) )</td>
<td>( X(k) \pm Y(k) )</td>
</tr>
<tr>
<td>( \lambda x(t) )</td>
<td>( \lambda X(k) )</td>
</tr>
<tr>
<td>( x(t) y(t) )</td>
<td>( \sum_{r=0}^{k} X(r) Y(k-r) )</td>
</tr>
<tr>
<td>( x(t) y(t) z(t) )</td>
<td>( \sum_{r=0}^{k} \sum_{l=0}^{r} X(l) Y(r-l) Z(k-r) )</td>
</tr>
<tr>
<td>( \frac{d^r}{dt^r} x(t) )</td>
<td>( \frac{(k+r)!}{k!} X(k+r) )</td>
</tr>
</tbody>
</table>

### Table 2. Transformed function of some nonlinear functions.

<table>
<thead>
<tr>
<th>Original function ( f(y(t)) )</th>
<th>Transformed function ( F(k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^{y(t)} )</td>
<td>( e^{Y(t_0)} \sum_{r=0}^{k} \sum_{m=0}^{r} (-1)^m Y^m(t_0) \left[ \frac{1}{k!} \frac{d^k}{dt^k} (y(t))^{r-m} \right]_{t=t_0} )</td>
</tr>
<tr>
<td>( \ln(y(t)) )</td>
<td>( \frac{1}{k!} \delta_k \ln(Y(t_0)) + \sum_{r=1}^{k} (-1)^{r-1} r Y^r(t_0) \left[ \frac{1}{k!} \frac{d^k}{dt^k} (y(t))^{r-m} \right]_{t=t_0} )</td>
</tr>
<tr>
<td>( \sin(y(t)) )</td>
<td>( \sum_{r=0}^{k} \frac{d^r}{dt^r} \sin(t) \sum_{r=0}^{k} \frac{d^r}{dt^r} \sin(t) \left[ \frac{1}{k!} \frac{d^k}{dt^k} (y(t))^{r-m} \right]_{t=t_0} )</td>
</tr>
<tr>
<td>( \cos(y(t)) )</td>
<td>( \sum_{r=0}^{k} \frac{d^r}{dt^r} \cos(t) \sum_{r=0}^{k} \frac{d^r}{dt^r} \cos(t) \left[ \frac{1}{k!} \frac{d^k}{dt^k} (y(t))^{r-m} \right]_{t=t_0} )</td>
</tr>
<tr>
<td>( \sinh(y(t)) )</td>
<td>( \sum_{r=0}^{k} \frac{d^r}{dt^r} \sinh(t) \sum_{r=0}^{k} \frac{d^r}{dt^r} \sinh(t) \left[ \frac{1}{k!} \frac{d^k}{dt^k} (y(t))^{r-m} \right]_{t=t_0} )</td>
</tr>
<tr>
<td>( \cosh(y(t)) )</td>
<td>( \sum_{r=0}^{k} \frac{d^r}{dt^r} \cosh(t) \sum_{r=0}^{k} \frac{d^r}{dt^r} \cosh(t) \left[ \frac{1}{k!} \frac{d^k}{dt^k} (y(t))^{r-m} \right]_{t=t_0} )</td>
</tr>
<tr>
<td>( \sqrt{y(t)} )</td>
<td>( \frac{\Gamma(k+\frac{1}{2})}{2\Gamma(k+1)\Gamma(\frac{1}{2})} \sum_{r=0}^{k} (-1)^r \binom{k}{r} (Y(t_0))^{\frac{1}{2}-r} \left[ \frac{d^k}{dt^k} (y(t))^r \right]_{t=t_0} )</td>
</tr>
<tr>
<td>( \frac{1}{y(t)} )</td>
<td>( (k+1) \sum_{r=0}^{k} \frac{(-1)^r}{(r+1)!} \binom{k}{r} (Y(t_0))^{-r-1} \left[ \frac{d^k}{dt^k} (y(t))^r \right]_{t=t_0} )</td>
</tr>
<tr>
<td>$t$</td>
<td>$x(t)$</td>
</tr>
<tr>
<td>-----</td>
<td>--------</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2231436</td>
</tr>
<tr>
<td>0.4</td>
<td>0.5108257</td>
</tr>
<tr>
<td>0.6</td>
<td>0.9162908</td>
</tr>
<tr>
<td>0.8</td>
<td>1.6094160</td>
</tr>
</tbody>
</table>

Table 3. The DTM and the MsDTM values are compared with the analytical solutions.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x(t)$</th>
<th>DTM</th>
<th>Analytical</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.2231436</td>
<td>0.2231436</td>
<td>0.2231436</td>
<td>0</td>
</tr>
<tr>
<td>0.4</td>
<td>0.5108248</td>
<td>0.5108248</td>
<td>0.5108248</td>
<td>0</td>
</tr>
<tr>
<td>0.6</td>
<td>0.9160622</td>
<td>0.9160622</td>
<td>0.9160622</td>
<td>0</td>
</tr>
<tr>
<td>0.8</td>
<td>1.5924103</td>
<td>1.5924103</td>
<td>1.5924103</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4. The DTM and the MsDTM values are compared with the analytical solutions.