**Linear dependence of four types of arithmetic functions**

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Linear dependence of four types of arithmetic functions

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Abstract

General criteria for the linear dependence of arithmetic functions over the complex field as well as several other criteria for arithmetic functions which are solutions of additive equation, multiplicative equation, exponential equation and logarithmic equation are derived. A number of examples are worked out in order to compare the results so obtained with the existing ones.

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1. Introduction

An arithmetic function is a complex-valued function defined over the set of natural numbers $\mathbb{N}$. Let $\mathcal{A}$ be a set of arithmetic functions equipped with the usual addition and the Dirichlet convolution defined for $f, g \in \mathcal{A}$ by

$$(f + g)(n) := f(n) + g(n), \quad (f * g)(n) := \sum_{d \mid n} f(d)g(n/d) \quad (n \in \mathbb{N}).$$

It is well-known by Cashwell and Everett (1959) that $(\mathcal{A}, +, *)$ is a unique factorization domain. The identity with respect to $*$ is the arithmetic function $I$ defined by $I(n) = 1$ for $n = 1$ and $I(n) = 0$ for $n > 1$. For $f \in \mathcal{A}$, its Dirichlet inverse, i.e., the inverse with respect to $*$, denoted by $f^{-1}$, exists if and only if $f(1) \neq 0$. A function $A \in \mathcal{A}$ is said to be additive if $A(m+n) = A(m) + A(n)$ $(m,n \in \mathbb{N})$. A function $M \in \mathcal{A}$ is said to be multiplicative if $M(mn) = M(m)M(n)$ for all $m,n \in \mathbb{N}$ with $\gcd(m,n) = 1$. An exponential function $E \in \mathcal{A}$ is a function satisfying $E(m+n) = E(m)E(n)$ $(m,n \in \mathbb{N})$. A logarithmic function $L \in \mathcal{A}$ is a function satisfying $L(mn) = L(m) + L(n)$ $(m,n \in \mathbb{N})$. 
In one of our previous works, the $\square$-linear dependence of three types of arithmetic functions, namely, additive, exponential and logarithmic has been investigated. It was found that

(i) two additive functions are always linearly dependent,
(ii) exponential functions are always linearly independent, and
(iii) logarithmic functions are linearly dependent if and only if they are algebraically dependent.

The case of multiplicative functions has previously been investigated by Kaczorowski, Molteni and Perelli (1999, 2006). They found that if the multiplicative functions $I, f_1, \ldots, f_n$ are pairwise non-equivalent (recall that two multiplicative arithmetic functions $f$ and $g$ are equivalent if $f(p^m) = g(p^m)$ for all $m \in \square$ and all but finitely many primes $p$), then $f_1, \ldots, f_n$ are $\square$-linearly independent.

Here, we continue our existing investigation. Complementing the results in Komatsu, Laohakosol and Reungsinsub (2011, 2012) we further investigate the $\square$-linear dependence of arithmetic functions which are solutions of additive equation, multiplicative equation, exponential equation and logarithmic equation. To this end, general criteria for linear dependence is proved. For additive functions, we extend one of our earlier results to cover the linearly dependence of general $n \geq 2$ functions. An alternative proof that exponential functions are always linearly independent is given. For multiplicative functions, conditions for a finite set of nonzero pairwise distinct multiplicative functions to be linearly independent are established. Conditions for linear independence of multiplicative functions based on an old method of Popken (1962) are proved. Finally, a necessary condition for linear independence of a finite set of nonzero pairwise distinct logarithmic arithmetic functions is derived. Several examples illustrating the so-obtained criteria are worked out in order to compare with the existing criteria.

2. Results

Our first result deals with two general criteria for linear (in)dependence.

**Theorem 1** Let $f_1, f_2, \ldots, f_n$ be $n \geq 2$ nonzero, pairwise distinct arithmetic functions. Assume that there exists an index $J \in \{1, \ldots, n\}$ such that $f_J(l) \neq 0$.

I. If there exist distinct $m_1, \ldots, m_{n-1} \in \square \setminus \{l\}$ such that
\[
\begin{vmatrix}
F_{i,1} & \cdots & F_{i-1,j} & F_{i+1,j} & \cdots & F_{n,1} \\
\vdots & & \vdots & \vdots & \ddots & \vdots \\
F_{i,n-1} & \cdots & F_{i-1,n} & F_{i+1,n} & \cdots & F_{n,n-1}
\end{vmatrix} \neq 0, \tag{1}
\]

where \( F_{i,j} = (f_i * f_j^{-1})(m_j) \) for \( i = 1, \ldots, n \) and \( j = 1, \ldots, n-1 \), then \( f_1, f_2, \ldots, f_n \) are \( \mathbb{F} \)-linearly independent.

II. If
\[
\begin{vmatrix}
F_{i,1} & \cdots & F_{i-1,j} & F_{i+1,j} & \cdots & F_{n,1} \\
\vdots & & \vdots & \vdots & \ddots & \vdots \\
F_{i,n-1} & \cdots & F_{i-1,n} & F_{i+1,n} & \cdots & F_{n,n-1}
\end{vmatrix} = 0, \tag{2}
\]

for all \( m_1, \ldots, m_{n-1} \in \mathbb{F} \setminus \{1\} \), then \( f_1, f_2, \ldots, f_n \) are \( \mathbb{F} \)-linearly dependent.

**Proof.** The proof can be found in Ponpetch, Laohakosol and Mavecha (2017).

2.1 Additive functions

In this subsection, we consider additive functions and start with an auxiliary result.

**Proposition 2** If \( A \in \mathcal{A} \) is a nonzero additive function, then its Dirichlet inverse is given by \( A^{-1} = \left( 1 / A^2(1) \right) \mu A. \)

**Theorem 3** If \( A_1, \ldots, A_n \) are \( n \geq 2 \) nonzero pairwise distinct additive arithmetic functions, then they are \( \mathbb{F} \)-linearly dependent.

**Proof.** The proof can be found in Ponpetch, Laohakosol and Mavecha (2017).

2.2 Exponential functions

In this section, we give another proof of the linear independence of exponential functions.

**Theorem 4** The nonzero exponential arithmetic functions \( E_1, \ldots, E_n \ (n \geq 2) \) are pairwise distinct if and only if they are \( \mathbb{F} \)-linearly independent.

**Proof.** We have proved in Ponpetch, Laohakosol and Mavecha (2017) that \( E_1, \ldots, E_n \) are linearly independent.
Conversely, assume that \( E_1, \ldots, E_n \) are not pairwise distinct. Then there are distinct indices \( i, j \in \{1, \ldots, n\} \) such that \( E_i(m) = E_j(m) \) \((m \in \square)\) yielding a linear relation.

2.3 Multiplicative functions

In this section, we present another condition for linear independence of multiplicative functions, and show that without such condition, there are examples of both dependent and independent functions.

**Theorem 5** Let \( M_1, M_2, \ldots, M_n \) \((n \geq 2)\) be nonzero, pairwise distinct multiplicative arithmetic functions. If there are distinct primes \( p_1, p_2, \ldots, p_{n-1} \) such that

\[
\left( M_i \ast M_j^{-1} \right)(p_1 p_2 \cdots p_{n-1}) \neq 0
\]

for all \( i, j \in \{1, \ldots, n\} \) with \( i \neq j \), then \( M_1, M_2, \ldots, M_n \) are \( \square \)-linearly independent.

**Proof.** We prove this theorem by induction on \( n \). For the case \( n = 2 \), suppose on the contrary that \( M_1, M_2 \) are \( \square \)-linearly dependent. Then there are complex constants \( c_1, c_2 \) not all zero such that

\[
c_1 M_1 + c_2 M_2 = 0. \tag{4}
\]

Operating by \( M_1^{-1} \) through (4), we get

\[
c_1 I(m) + c_2 \left( M_2 \ast M_1^{-1} \right)(m) = 0 \quad (m \in \square). \tag{5}
\]

Replacing \( m \) by \( p_1 \), the prime \( p_1 \) satisfying (3), in (5), we get \( c_2 \left( M_2 \ast M_1^{-1} \right)(p_1) = 0 \). Using (3), we deduce \( c_2 = 0 \). Putting \( c_2 = 0 \) in (4), we get \( c_1 = 0 \).

Assume now that the theorem holds up to \( n-1 \) functions, we prove its validity for \( n \) functions. Suppose on the contrary that \( M_1, M_2, \ldots, M_n \) \((n \geq 3)\) are \( \square \)-linearly dependent. Then there are complex constants \( c_1, c_2, \ldots, c_n \) not all zero such that

\[
c_1 M_1 + c_2 M_2 + \cdots + c_n M_n = 0. \tag{6}
\]

Operating by \( M_1^{-1} \) through (6), we get

\[
c_1 I(m) + c_2 F_2(m) + \cdots + c_n F_n(m) = 0 \quad (m \in \square), \tag{7}
\]
where $F_j = M_j * M_i^{-1}$ \((j = 2, \ldots, n)\). Let \(p_1, \ldots, p_{n-1}\) be distinct primes satisfying (3) and let

\[
V_{p_{n-1}} = \{m \in \mathbb{D} : \gcd(m, p_{n-1}) = 1\}.
\]

Replacing \(m\) by \(t p_{n-1}, t \in V_{p_{n-1}}\) in (7), we get

\[
c_2 F_2(p_{n-1}) F_2(t) + \cdots + c_n F_n(p_{n-1}) F_n(t) = 0. \tag{8}
\]

For \(j = 2, \ldots, n\), define

\[
G_j(m) = \begin{cases} F_j(m) & \text{if } m \in V_{p_{n-1}} \\ 0 & \text{otherwise}. \end{cases}
\]

It is easy to check each \(G_j\) is multiplicative and so that the relation (8) becomes

\[
d_2 G_2(m) + \cdots + d_n G_n(m) = 0 \quad (m \in \mathbb{D}), \tag{9}
\]

where \(d_j = c_j F_j(p_{n-1}) (j = 2, \ldots, n)\). If \(G_j (j = 2, \ldots, n)\) are zero functions, then \(F_j(m) = 0\) for \(m \in V_{p_{n-1}}\). Since \(p_1 \in V_{p_{n-1}}\), using the multiplicativity of \(M_j * M_i^{-1}\) and (3), we get \(0 = F_j(p_1) = (M_j * M_i^{-1})(p_1) \neq 0\), which is a contradiction. Thus \(G_j (j = 2, \ldots, n)\) are nonzero functions. Since \(0 \neq (M_j * M_k^{-1})(p_1) = M_j(p_1) - M_k(p_1)\) for \(j\) not equal to \(k\), we have

\[
G_j(p_1) = F_j(p_1) = (M_j * M_i^{-1})(p_1) = M_j(p_1) - M_i(p_1)
\]

\[
\neq M_k(p_1) - M_i(p_1) = (M_k * M_i^{-1})(p_1) = F_k(p_1) = G_k(p_1)
\]

showing that \(G_j (j = 2, \ldots, n)\) are pairwise distinct. Since

\[
(G_j * G_k^{-1})(p_1 \ldots p_{n-2}) = (G_j * G_k^{-1})(p_1) \cdots (G_j * G_k^{-1})(p_{n-2})
\]

\[
= (F_j * F_k^{-1})(p_1) L (F_j * F_k^{-1})(p_{n-2})
\]

\[
= (F_j(p_1) - F_k(p_1)) \cdots (F_j(p_{n-2}) - F_k(p_{n-2}))
\]

\[
= (M_j(p_1) - M_k(p_1)) \cdots (M_j(p_{n-2}) - M_k(p_{n-2}))
\]

\[
\neq (M_j * M_k^{-1})(p_1) \cdots (M_j * M_k^{-1})(p_{n-2}) \neq 0,
\]

for all \(j, k \in \{2, \ldots, n\}, j \neq k\), the multiplicative functions \(G_j (j = 2, \ldots, n)\) satisfying (3). Thus, the induction hypothesis yields that \(G_2, \ldots, G_n\) are \(\mathbb{D}\)-linearly independent,
which in turn implies, from (9), that $0 = d_j = c_j F_j(p_{m_j}) (j = 2, \ldots, n)$. Since $F_j = M_j \ast M_j^{-1}$, using the multiplicativity and (3), we have $F_j(p_{m_j}) \neq 0$, which shows that $c_j = 0 (j = 2, \ldots, n)$. Replacing $c_j = 0 (j = 2, \ldots, n)$ in (6), we get $c_1 = 0$.

If the condition (3) does not hold, then $M_1, M_2, \ldots, M_n$ can either be linearly dependent, or independent as seen from the following examples.

**Example 6.** Consider the four functions $F_1, F_2, F_3, F_4$, defined, respectively, by

$F_1(1) = F_1(2) = 1$, $F_1(3) = F_1(5) = F_1(6) = F_1(10) = 2$, $F_1(15) = F_1(30) = 4$, $F_1(n) = 0$ for all positive integers $n \neq 1, 2, 3, 5, 6, 10, 15, 30$.

$F_2(1) = F_2(3) = 1$, $F_2(2) = F_2(5) = F_2(6) = F_2(15) = 2$, $F_2(10) = F_2(30) = 4$, $F_2(n) = 0$ for all positive integers $n \neq 1, 2, 3, 5, 6, 10, 15, 30$.

$F_3(1) = 1$, $F_3(2) = 3$, $F_3(3) = F_3(5) = 2$, $F_3(6) = F_3(10) = 6$, $F_3(15) = 4$, $F_3(30) = 12$, $F_3(n) = 0$ for all positive integers $n \neq 1, 2, 3, 5, 6, 10, 15, 30$.

$F_4(1) = 1$, $F_4(2) = F_4(5) = 2$, $F_4(3) = 3$, $F_4(6) = F_4(15) = 6$, $F_4(10) = 4$, $F_4(30) = 12$, $F_4(n) = 0$ for all positive integers $n \neq 1, 2, 3, 5, 6, 10, 15, 30$.

It is easily checked that $F_1, F_2, F_3, F_4$ are multiplicative functions with inverses, for $r \in \mathbb{Q}$,

$F_1^{-1}(1) = 1$, $F_1^{-1}(2^r) = (-1)^r$, $F_1^{-1}(3^r) = F_1^{-1}(5^r) = (-2)^r$, $F_1^{-1}(6^r) = F_1^{-1}(10^r) = (2)^r$, $F_1^{-1}(15^r) = (4)^r$, $F_1^{-1}(30^r) = (-4)^r$, $F_1^{-1}(n) = 0$ for all positive integers $n \neq 1, 2^r$, $3^r, 5^r, 6^r, 10^r, 15^r, 30^r$.

$F_2^{-1}(1) = 1$, $F_2^{-1}(2^r) = F_2^{-1}(5^r) = (-2)^r$, $F_2^{-1}(3^r) = (-1)^r$, $F_2^{-1}(6^r) = F_2^{-1}(15^r) = (2)^r$, $F_2^{-1}(10^r) = (4)^r$, $F_2^{-1}(30^r) = (-4)^r$, $F_2^{-1}(n) = 0$ for all positive integers $n \neq 1, 2^r$, $3^r, 5^r, 6^r, 10^r, 15^r, 30^r$.

$F_3^{-1}(1) = 1$, $F_3^{-1}(2^r) = (-3)^r$, $F_3^{-1}(3^r) = F_3^{-1}(5^r) = (-2)^r$, $F_3^{-1}(6^r) = F_3^{-1}(10^r) = (6)^r$, $F_3^{-1}(15^r) = (4)^r$, $F_3^{-1}(30^r) = (-12)^r$, $F_3^{-1}(n) = 0$ for all positive integers $n \neq 1, 2^r$, $3^r, 5^r, 6^r, 10^r, 15^r, 30^r$.

$F_4^{-1}(1) = 1$, $F_4^{-1}(2^r) = F_4^{-1}(5^r) = (-2)^r$, $F_4^{-1}(3^r) = (-3)^r$, $F_4^{-1}(6^r) = F_4^{-1}(15^r) = (6)^r$, $F_4^{-1}(10^r) = (4)^r$, $F_4^{-1}(30^r) = (-12)^r$, $F_4^{-1}(n) = 0$ for all positive integers $n \neq 1, 2^r$, $3^r, 5^r, 6^r, 10^r, 15^r, 30^r$. 
In this case, the condition (3) does not hold while we have -linearly dependent relation \( F_1 - F_2 + F_3 - F_4 = 0 \).

**Example 7.** Consider the four functions \( G_1, G_2, G_3, G_4 \), defined, respectively, by

\[
G_1(n) = 1, \quad G_2(n) = G_1(5) = 2, \quad G_3(n) = G_1(10) = 4, \quad G_4(n) = G_1(15) = 8, \quad G_1(n) = 0 \text{ for all positive integers } n \neq 1, 2, 3, 5, 6, 10, 15, 30.
\]

\[
G_2(n) = 1, \quad G_2(2) = 3, \quad G_2(3) = 9, \quad G_2(5) = 2, \quad G_2(6) = 27, \quad G_2(10) = 6, \quad G_2(15) = 18, \quad G_2(30) = 54, \quad G_2(n) = 0 \text{ for all positive integers } n \neq 1, 2, 3, 5, 6, 10, 15, 30.
\]

\[
G_3(n) = 1, \quad G_3(2) = 5, \quad G_3(3) = 25, \quad G_3(5) = 2, \quad G_3(6) = 125, \quad G_3(10) = 10, \quad G_3(15) = 50, \quad G_3(30) = 250, \quad G_3(n) = 0 \text{ for all positive integers } n \neq 1, 2, 3, 5, 6, 10, 15, 30.
\]

\[
G_4(n) = 1, \quad G_4(2) = 7, \quad G_4(3) = 49, \quad G_4(5) = 2, \quad G_4(6) = 343, \quad G_4(10) = 14, \quad G_4(15) = 98, \quad G_4(30) = 686, \quad G_4(n) = 0 \text{ for all positive integers } n \neq 1, 2, 3, 5, 6, 10, 15, 30.
\]

It is easily checked that \( G_1, G_2, G_3, G_4 \) are multiplicative functions with inverses, for \( r \in \mathbb{R} \),

\[
G_1^{-1}(1) = 1, \quad G_1^{-1}(2^r) = G_1^{-1}(5^r) = (-2)^r, \quad G_1^{-1}(3^r) = (-4)^r, \quad G_1^{-1}(6^r) = G_1^{-1}(15^r) = (8)^r, \quad G_1^{-1}(10^r) = (4)^r, \quad G_1^{-1}(30^r) = (-16)^r, \quad G_1^{-1}(n) = 0 \text{ for all positive integers } n \neq 1, 2^r, 3^r, 5^r, 6^r, 10^r, 15^r, 30^r.
\]

\[
G_2^{-1}(1) = 1, \quad G_2^{-1}(2^r) = (-3)^r, \quad G_2^{-1}(3^r) = (-9)^r, \quad G_2^{-1}(5^r) = (-2)^r, \quad G_2^{-1}(6^r) = (27)^r, \quad G_2^{-1}(10^r) = (6)^r, \quad G_2^{-1}(15^r) = (18)^r, \quad G_2^{-1}(30^r) = (-54)^r, \quad G_2^{-1}(n) = 0 \text{ for all positive integers } n \neq 1, 2^r, 3^r, 5^r, 6^r, 10^r, 15^r, 30^r.
\]

\[
G_3^{-1}(1) = 1, \quad G_3^{-1}(2^r) = (-5)^r, \quad G_3^{-1}(3^r) = (-25)^r, \quad G_3^{-1}(5^r) = (-2)^r, \quad G_3^{-1}(6^r) = (125)^r, \quad G_3^{-1}(10^r) = (10)^r, \quad G_3^{-1}(15^r) = (50)^r, \quad G_3^{-1}(30^r) = (-250)^r, \quad G_3^{-1}(n) = 0 \text{ for all positive integers } n \neq 1, 2^r, 3^r, 5^r, 6^r, 10^r, 15^r, 30^r.
\]

\[
G_4^{-1}(1) = 1, \quad G_4^{-1}(2^r) = (-7)^r, \quad G_4^{-1}(3^r) = (-49)^r, \quad G_4^{-1}(5^r) = (-2)^r, \quad G_4^{-1}(6^r) = (343)^r, \quad G_4^{-1}(10^r) = (14)^r, \quad G_4^{-1}(15^r) = (98)^r, \quad G_4^{-1}(30^r) = (-686)^r, \quad G_4^{-1}(n) = 0 \text{ for all positive integers } n \neq 1, 2^r, 3^r, 5^r, 6^r, 10^r, 15^r, 30^r.
\]

Here, the condition (3) does not hold. We show that \( G_1, G_2, G_3, G_4 \) are -linearly independent. Suppose on the contrary that \( G_1, G_2, G_3, G_4 \) are -linearly dependent.

Then there are complex constants \( c_1, \ldots, c_4 \) not all zero such that
Putting $n = 1, 2, 3, 6$, respectively in (10), we get

\[ c_1 G_1(n) + c_2 G_2(n) + c_3 G_3(n) + c_4 G_4(n) = 0. \] (10)

Using the defining values of $G_1, G_2, G_3, G_4$, the coefficient matrix of the above system is non-singular, and this implies that $c_1 = c_2 = c_3 = c_4 = 0$, which is a contradiction.

We proceed now to use a method of Popken (1962) to derive criterion for linear independence of multiplicative functions. Let $S \subseteq \mathbb{N}$ be a commutative semi-group in which a unique factorization condition holds. We assume that $S$ has an identity-element 1 and no other unit than 1. By a reduced semi-group $S_0$, we mean a set of $m \in S$ such that $\gcd(m, x_0) = 1$ for some fixed $x_0 \in S$.

**Theorem 8** Let $M_1, M_2, \ldots, M_n$ ($n \geq 2$) be nonzero, pairwise distinct multiplicative arithmetic functions. Suppose that there exists a semi-group $S \subseteq \mathbb{N}$ with the properties described above, and there are $c_1(\neq 0), c_2, \ldots, c_n \in S$ such that

\[ c_1 M_1(t) + \cdots + c_n M_n(t) = 0 \quad (t \in S). \] (11)

Then there is at least one suffix $h \in \{2, \ldots, n\}$ such that

\[ M_i(m) = M_h(m) \quad (m \in S_0) \] (12)

for some reduced semi-group $S_0 \subseteq S$. Moreover, $M_i * M^{-1}_h$ vanishes on some reduced semi-group $S_0 \setminus \{1\}$ contained in $S$.

**Proof.** We prove by induction on $n$. For the case $n = 2$, there are $c_1(\neq 0), c_2 \in S$ such that

\[ c_1 M_1(t) + c_2 M_2(t) = 0 \quad (t \in S). \] (13)

Taking $t = 1$ in (13), we get $c_1 = -c_2 \neq 0$. Thus, (13) yields $M_1(t) = M_2(t)$ for all $t \in S$, i.e., (12) holds with $h = 2$ and $S_0 = S$. Assume that the theorem holds up to $n - 1$ functions, we next prove its validity for $n$ functions. If $M_i(t) = M_h(t)$ for all $t \in S$,
then there is nothing more to prove. Otherwise, \( M_1 \neq M_n \) on \( S \), and so there exists \( x_i \in S \) such that \( M_1(x_i) \neq M_n(x_i) \). By assumption, we have
\[
c_i M_1(t) + \cdots + c_n M_n(t) = 0 \quad (t \in S).
\] (14)

Let \( S_i = \{ m \in S ; \gcd(m, x_i) = 1 \} \subseteq S \). Taking \( t = mx_i \), where \( m \in S_i \), in (14) and by multiplicity, we get
\[
c_i M_1(x_i)M_1(m) + \cdots + c_n M_n(x_i)M_n(m) = 0 \quad (m \in S_i).
\] (15)

Taking \( t = m \), where \( m \in S_i \), in (14) and multiplying by \( M_n(x_i) \), we have
\[
c_i M_1(x_i)M_1(m) + \cdots + c_n M_n(x_i)M_n(m) = 0 \quad (m \in S_i).
\] (16)

Subtracting (15) and (16), we have
\[
c_i (M_1(x_i) - M_n(x_i))M_1(m) + \cdots + c_{n-1} (M_{n-1}(x_i) - M_n(x_i))M_{n-1}(m) = 0 \quad (m \in S_i).
\] (17)

Since \( c_i (M_1(x_i) - M_n(x_i)) \neq 0 \), the relation (17) is similar to the relation (11) on \( S \). By the induction hypothesis, there exists a reduced semi-group \( S_2 \subseteq S_i \subseteq S \) such that
\( M_1 = M_h \) on \( S_2 \) for some \( h \in \{ 2, \ldots, n-1 \} \). Define \( \phi = M_1 * M_h^{-1} \). Clearly
\[
\phi(m) = (M_1 * M_h^{-1})(m) = \sum_{d|m} M_1(d)M_h^{-1}\left(\frac{m}{d}\right)
\] (18)
for all \( m \in S_0 \setminus \{1\} \). Since \( \gcd(m, x_o) = 1 \), we have \( \gcd(d, x_o) = 1 \) and \( \gcd(m/d, x_o) = 1 \), i.e., \( d, m/d \in S_0 \). The relation (18) becomes
\[
\phi(m) = (M_1 * M_h^{-1})(m) = \sum_{d|m} M_1(d)M_h^{-1}\left(\frac{m}{d}\right) = \sum_{d|m} M_h(d)M_h^{-1}\left(\frac{m}{d}\right) = I(m) = 0
\]
which holds for all \( m \in S_0 \setminus \{1\} \).

**Corollary 9** Let \( M_1, M_2, \ldots, M_n \) \( (n \geq 2) \) be nonzero, pairwise distinct multiplicative arithmetic functions. For all indices \( j = 2, \ldots, n \), if there exists an \( \alpha \in \mathbb{N} \) such that the relation
\[
(M_1 * M_j^{-1})(p^\alpha) \neq 0
\] (19)
holds for all primes $p$, then $M_1, M_2, \ldots, M_n$ are $-\text{linearly independent}$.

**Proof.** Suppose by the contrary $M_1, M_2, \ldots, M_n$ are $-\text{linearly dependent}$. Then there are complex constants $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1 M_1(t) + c_2 M_2(t) + \cdots + c_n M_n(t) = 0 \ (t \in \mathbb{I}). \ (20)$$

Since $S \subseteq \mathbb{I}$, the relation (20) restricts to

$$c_1 M_1(m) + c_2 M_2(m) + \cdots + c_n M_n(m) = 0 \ (m \in S). \ (21)$$

Without loss of generality, assume that $c_1 \neq 0$. By Theorem 8, for some $j = 2, \ldots, n$, the relation $M_j(m) = M_j(m)$ holds for all $m$ in some reduced semi-group

$$S_0 = \{m \in S; \gcd(m, x_0) = 1\} \subseteq S, x_0 \in S, \text{ and } M_j M_j^{-1} \text{ vanishes on } S_0 \setminus \{1\}. \text{ Let } x_0 = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \in S \subseteq \mathbb{I}, p_1, \ldots, p_r \text{ being primes, and } \beta_1, \ldots, \beta_r \in \mathbb{I}. \text{ Choose another prime } q \not\in \{p_1, \ldots, p_r\}. \text{ Then for any } \alpha \in \mathbb{I}, \text{ we have } \gcd(q^{\alpha}, x_0) = 1, \text{ i.e., } q^{\alpha} \in S_0. \text{ Thus, } \left(M_j M_j^{-1}(q^{\alpha})\right) = 0, \text{ which is a contradiction.}

We next exhibit by examples that Theorem 5, the result in Kaczorowski, Molteni and Perelli (2006) and Corollary 9 are somewhat independent of one another by analyzing the case of two multiplicative arithmetic functions $f$ and $g$. In this case, the corresponding three linearly independent conditions are:

- **A.** (Theorem 5) $f$ and $g$ are nonzero pairwise distinct with $(f * g^{-1})(p) \neq 0$ for some prime $p$.
- **B.** (Kaczorowski, Molteni and Perelli 2006) $f, g$ are pairwise non-equivalent, i.e., there are infinitely many primes $p$ such that

$$I(p^m) \neq f(p^m), \ I(p^m) \neq g(p^m) \text{ and } f(p^m) \neq g(p^m) \text{ for some } m \in \mathbb{I}.

- **C.** (Corollary 9) $f$ and $g$ are nonzero pairwise distinct with $(f * g^{-1})(p^m) \neq 0$ for all primes $p$.

**Example 10** Consider two multiplicative functions $f_i$ and $g_i$ defined by

$$f_i(n) = n, \ g_i(n) = n^2 \ (n \in \mathbb{I}).$$

- **A** is true because $(f_1 * g_1^{-1})(p) = p - p^2 \neq 0$ for all primes $p$.
- **B** is true because

$$I(p^m) = 0 \neq p^m = f_1(p^m), \ I(p^m) = 0 \neq p^2m = g_1(p^m), \ f_1(p^m) = p^m \neq p^2m = g_1(p^m) \text{ for all primes } p \text{ and for all } m \in \mathbb{I}.$$
• $C_1$ is true because $(f_i \ast g_i^{-1})(p) = p - p^2 \neq 0$ for all primes $p$.

The functions $f_i$ and $g_i$ are \( \square \)-linearly independent by any one of the three criteria $A$, $B$ or $C_1$.

**Example 11** Consider two multiplicative functions $f_2$ and $g_2$ defined by

\[ f_2(1) = 1, \quad f_2(2) = 3, \quad f_2(n) = 0 \text{ for } n \neq 1, 2 \quad \text{and} \quad g_2(1) = 1, \quad g_2(2) = 5, \quad g_2(n) = 0 \text{ for } n \neq 1, 2. \]

• $A$ is true because evaluating at the prime 2, we get $(f_2 \ast g_2^{-1})(2) = 3 - 5 = -2 \neq 0$.

• $B$ does not hold because $f_2(p^m) = g_2(p^m) = 0$ for all primes $p \neq 2$ and for all $m \in \square$.

• $C_\alpha$ does not hold for all $\alpha \in \square$ because $(f_2 \ast g_2^{-1})(p^\alpha) = 0$ for all primes $p \neq 2$.

Thus, $f_2$ and $g_2$ are \( \square \)-linearly independent by the criterion $A$, but not by the other two criteria.

**Example 12** Consider two multiplicative functions $f_3$ and $g_3$ defined by

\[ f_3(1) = 1, \quad f_3(p) = 0, \quad f_3(2^m) = 0 \text{ for } m \geq 2, \quad f_3(p^m) = p^m \text{ for } p \neq 2, m \geq 2, \]

\[ g_3(1) = 1, \quad g_3(p) = 0, \quad g_3(2^m) = 0 \text{ for } m \geq 2, \quad g_3(p^m) = p^{2m} \text{ for } p \neq 2, m \geq 2. \]

• $A$ does not hold because $(f_3 \ast g_3^{-1})(p) = 0$ for all primes $p$.

• $B$ is true because

\[ I(p^m) = 0 \neq p^m = f_3(p^m), \quad I(p^m) = 0 \neq p^{2m} = g_3(p^m), \quad f_3(p^m) = p^m \neq p^{2m} = g_3(p^m) \]

for all primes $p \neq 2$ and for all $m \in \square \setminus \{1\}$.

• $C_\alpha$ does not hold for all $\alpha \in \square$ because $(f_3 \ast g_3^{-1})(2^\alpha) = 0$.

Thus, $f_3$ and $g_3$ are \( \square \)-linearly independent by the criterion $B$, but not by the other two criteria.

**Example 13** Consider two multiplicative functions $f_4$ and $g_4$ defined by

\[ f_4(1) = 1, \quad f_4(p) = 0, \quad f_4(2^m) = 2 \text{ for } m \geq 2, \quad f_4(p^m) = p^{2m} \text{ for } p \neq 2, m \geq 2, \]

\[ g_4(1) = 1, \quad g_4(p) = 0, \quad g_4(2^m) = 4 \text{ for } m \geq 2, \quad g_4(p^m) = 0 \text{ for } p \neq 2, m \geq 2. \]

• $A$ does not hold because $(f_4 \ast g_4^{-1})(p) = 0 - 0 = 0$ for all primes $p$.

• $B$ does not hold because $I(p^m) = g_4(p^m) = 0$ for all primes $p \neq 2$ and for all $m \in \square$. 
• $C_3$ is true because
\[ (f_4 \ast g_4^{-1})(p^2) = -g_4(p^2) + g_4^2(p) - f_4(p)g_4(p) + f_4(p^2) = 0 + 0 - 0 + p^4 \neq 0 \]
for all primes $p \neq 2$ and $(f_4 \ast g_4^{-1})(2^2) = -4 + 0 - 0 + 2 = -2 \neq 0$.

Thus, $f_4$ and $g_4$ are $\square$-linearly independent by the criterion $C_2$, but not by the other two criteria.

**Example 14** Consider two multiplicative functions $f_5$ and $g_5$ defined by
\[ f_5(1) = 1, \quad f_5(2) = 2, \quad f_5(p) = 0 \ (p \neq 2), \quad f_5(2^m) = 0 \ (m \geq 2), \quad f_5(3^m) = 0 \ (m \geq 2), \]
\[ f_5(p^m) = p^{2m} \ (p \neq 2, 3; m \geq 2), \quad g_5(1) = 1, \quad g_5(2) = 4, \quad g_5(p) = 0 \ (p \neq 2), \quad g_5(2^m) = 0 \ (m \geq 2), \quad g_5(3^m) = 0 \ (m \geq 2), \]
\[ g_5(p^m) = p^{2m} \ (p \neq 2, 3; m \geq 2). \]

• $A$ is true because there is a prime 2 such that $(f_5 \ast g_5^{-1})(2) = 2 - 4 = -2 \neq 0$.
• $B$ is true because
\[ I(p^m) = 0 \neq p^m = f_5(p^m), \quad I(p^m) = 0 \neq p^{2m} = g_5(p^m), \quad f_5(p^m) = p^m \neq p^{2m} = g_5(p^m) \]
for all primes $p \neq 2, 3$ and for all $m \in \mathbb{Q} \setminus \{1\}$.
• $C_a$ does not hold for all $\alpha \in \mathbb{N}$ because $(f_5 \ast g_5^{-1})(3^\alpha) = 0$.

Thus, $f_5$ and $g_5$ are $\square$-linearly independent by the criterion $A$ and $B$, but not by $C_a$.

**Example 15** Consider two multiplicative functions $f_6$ and $g_6$ defined by
\[ f_6(1) = 1, \quad f_6(p) = 0 \ (p \neq 2), \quad f_6(2^m) = 2 \ (m \geq 1), \quad f_6(p^m) = p^{2m} \ (p \neq 2, m \geq 2), \]
\[ g_6(1) = 1, \quad g_6(p) = 0 \ (p \neq 2), \quad g_6(2^m) = 4 \ (m \geq 1), \quad g_6(p^m) = 0 \ (p \neq 2, m \geq 2). \]

• $A$ is true because there is a prime 2 such that $(f_6 \ast g_6^{-1})(2) = 2 - 4 = -2 \neq 0$.
• $B$ does not hold because $I(p^m) = g_6(p^m) = 0$ for all primes $p \neq 2$ and for all $m \in \mathbb{N}$.
• $C_2$ is true because
\[ (f_6 \ast g_6^{-1})(p^2) = -g_6(p^2) + g_6^2(p) - f_6(p)g_6(p) + f_6(p^2) = 0 + 0 - 0 + p^4 \neq 0 \]
for all primes $p \neq 2$ and $(f_6 \ast g_6^{-1})(2^2) = -4 + 4^2 - 2(4) + 2 = 6 \neq 0$.

Thus, $f_6$ and $g_6$ are $\square$-linearly independent by the criterion $A$ and $C_2$, but not by $B$.  

For Proof Read only
Example 16 Consider two multiplicative functions \( f \) and \( g \) defined by

\[
\begin{align*}
\ell_1 = 1, & \quad \ell_2 = 2, & \quad \ell_3(p) = 0 \ (p \not= 2), & \quad \ell_4 = 2 \ (m \geq 2), \quad \ell_5(p^m) = p^m \ (p \not= 2, m \geq 2), \\
\ell_6 = 1, & \quad \ell_7 = 2, & \quad \ell_8(p) = 0 \ (p \not= 2), & \quad \ell_9 = 2 \ (m \geq 2), \quad \ell_{10}(p^m) = p^{2m} \ (p \not= 2, m \geq 2).
\end{align*}
\]

- A does not hold because \((f \ast g^{-1})(p) = 0\) for all primes \( p \).
- B is true because
  \[
  \begin{align*}
  I(p^m) = 0 & \neq p^m = f_1(p^m), & \quad I(p^m) = 0 & \neq p^{2m} = g_1(p^m), & \quad f_1(p^m) = p^m & \neq p^{2m} = g_1(p^m)
  \end{align*}
  \]
  for all primes \( p \not= 2 \) and for all \( m \in \mathbb{N} \setminus \{1\} \).
- C is true because
  \[
  \begin{align*}
  (f \ast g^{-1})(p^3) &= f_1(p^3) - f_1(p^2)g_1(p) - f_1(p)g_1(p^2) + f_1(p)g_1^2(p) - g_1(p^3) \\
  &= 2g_1(p^2)g_1(p) - g_1^3(p) \\
  &= p^3 - p^6 \not= 0
  \end{align*}
  \]
  for all primes \( p \not= 2 \) and \((f \ast g^{-1})(2^3) = 2 - 4 - 0 + 8 - 0 + 8 = -2 \not= 0\).

Thus, \( f \) and \( g \) are \( \mathbb{N} \)-linearly independent by the criterion B and C, but not by A.

2.4 Logarithmic functions

As for logarithmic functions, we show that subject to an extra condition, they are \( \mathbb{N} \)-linearly independent, while without such a condition they are \( \mathbb{N} \)-linearly dependent.

Theorem 17 Let \( L_1, \ldots, L_n \) be \( n \geq 2 \) nonzero pairwise distinct logarithmic arithmetic functions.

I. If there exist distinct primes \( p_1, \ldots, p_n \) such that

\[
\begin{vmatrix}
L_1(p_1) & L_2(p_1) & \cdots & L_n(p_1) \\
\vdots & & & \\
L_1(p_n) & L_2(p_n) & \cdots & L_n(p_n)
\end{vmatrix} \neq 0, \quad (22)
\]

then \( L_1, \ldots, L_n \) are \( \mathbb{N} \)-linearly independent.

II. If the condition

\[
\begin{vmatrix}
L_1(p_1) & L_2(p_1) & \cdots & L_n(p_1) \\
\vdots & & & \\
L_1(p_n) & L_2(p_n) & \cdots & L_n(p_n)
\end{vmatrix} = 0 \quad (23)
\]

then \( L_1, \ldots, L_n \) are \( \mathbb{N} \)-linearly dependent.
holds for all distinct primes $p_1, \ldots, p_n$, then $L_1, \ldots, L_n$ are $\square$-linearly dependent.

**Proof.** I. The proof can be found in Ponpetch, Laohakosol and Mavecha (2017).

II. We first treat the case $n = 2$. From (23) we have

$$c_1 L_1(p_1) + c_2 L_2(p_1) = 0, \ c_1 L_1(p_2) + c_2 L_2(p_2) = 0$$

(24)

for some complex constants $c_1, c_2$ not all zero. Without loss of generality, assume $c_1 \neq 0$. Then the system (24) becomes

$$L_1(p_1) = d_2 L_2(p_1), \ L_1(p_2) = d_2 L_2(p_2)$$

(25)

where $d_2 = -c_2 / c_1 \in \square$. Taking another prime $p_j$ in place of $p_1$ in (23), we get another system

$$c'_1 L_1(p_j) + c'_2 L_2(p_j) = 0, \ c'_1 L_1(p_2) + c'_2 L_2(p_2) = 0.$$  

(26)

If $c'_1 = 0$, then $L_2 = 0$, which is a contradiction. Thus $c'_1 \neq 0$, and rewrite (26) as

$$L_1(p_j) = d'_2 L_2(p_j), \ L_1(p_2) = d'_2 L_2(p_2).$$

(27)

Subtracting corresponding equations (except the first) in the two systems (25) and (27), we get $d_2 = d'_2$. Hence $L_i(p) = d_2 L_2(p)$ for all prime $p_i$ implying that $L_i = d_2 L_2$. Now, we proceed to the general case. Assume the result holds up to $n-1$ functions, we use induction to show that it holds for $n$ functions. The vanishing of the determinant (23) infers that their columns are dependent, i.e., there are complex constants $c_1, \ldots, c_n$ not all zero such that

$$c_1 L_1(p_1) + c_2 L_2(p_1) + \cdots + c_n L_n(p_1) = 0, \ldots, c_1 L_1(p_n) + c_2 L_2(p_n) + \cdots + c_n L_n(p_n) = 0.$$  

(28)

Since not all of $c_1, \ldots, c_n$ are zero, without loss of generality, assume $c_1 \neq 0$. The system (28) becomes

$$L_1(p_1) = d_2 L_2(p_1) + \cdots + d_n L_n(p_1), \ldots, L_1(p_n) = d_2 L_2(p_n) + \cdots + d_n L_n(p_n),$$

(29)

where $d_i = -c_i / c_1 \in \square$ ($i = 2, \ldots, n$). Taking another prime $p_j$ in place of $p_i$ in (28), we get another system

$$c'_1 L_1(p_j) + c'_2 L_2(p_j) + \cdots + c'_n L_n(p_j) = 0, \ldots, c'_1 L_1(p_n) + c'_2 L_2(p_n) + \cdots + c'_n L_n(p_n) = 0.$$  

(30)

If $c'_i = 0$, from the system (30) leaving the first row we get another homogeneous system of order $n-1$. If the determinant of the system is 0, we are done by the
induction hypothesis; otherwise it implies that $c'_2 = \cdots = c'_n = 0$, which is a contradiction. If $c'_1 \neq 0$, rewrite (30) as

$$L_i(p_j) = d'_i L_2(p_j) + \cdots + d'_n L_n(p_j), \ldots, L_i(p_n) = d'_i L_2(p_n) + \cdots + d'_n L_n(p_n).$$

(31)

where $d'_i = -c'_i / c'_1 \in \square (i = 2, \ldots, n)$. Subtracting corresponding equations (except the first) in the two systems (29) and (31) leads to the homogeneous system

$$(d_2 - d'_2)L_2(p_2) + (d_3 - d'_3)L_2(p_2) + \cdots + (d_n - d'_n)L_n(p_2) = 0, \ldots,$n

$$(d_2 - d'_2)L_2(p_n) + (d_3 - d'_3)L_2(p_n) + \cdots + (d_n - d'_n)L_n(p_n) = 0.$n

If the coefficient matrix of this last system is singular, we return to the lower case. If it is non-singular, then $d_i = d'_i (i = 2, \ldots, n)$, implying that $L_i(p) = d_2 L_2(p) + \cdots + d_n L_n(p)$ for all prime $p$, and so $L_i = d_2 L_2 + \cdots + d_n L_n$.

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