Some Factorizations of Ramanujan’s Cubic Continued Fraction

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Original Article

Some Factorizations of Ramanujan’s Cubic Continued Fraction

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Abstract

We derive new identities involving Ramanujan’s cubic continued fraction which are analogous to those of the famous Rogers-Ramanujan continued fraction. Using such new identities enables us to give the new proof of several early results of this continued fraction.

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1. Introduction

Ramanujan’s general theta-function $f(a, b)$ is defined by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{(n+1)/2} b^{n(n-1)/2},$$

where $a, b$ are complex numbers with $|ab| < 1$.

As customary and throughout this paper, we assume that $q$ is a complex number with $|q| < 1$ and use the standard notation

$$(a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k) \quad \text{and} \quad (a; q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n).$$
The function $f(a,b)$ can be written in terms of infinite products via Jacobi’s triple product identity (Berndt, 1991) given by

$$f(a,b) = \frac{(-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty}{(a;ab)_\infty (b;ab)_\infty (ab;ab)_\infty}, \quad (1.1)$$

Define

$$f(q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n+1)/2} = (q;q)_\infty, \quad (1.2)$$

$$\varphi(q) := f(-q,-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{f^2(-q)}{f(-q^2)}, \quad (1.3)$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{f^2(-q^2)}{f(-q)}, \quad (1.4)$$

$$\chi(q) := \frac{f(-q)}{f(-q^2)}. \quad (1.5)$$

The last equalities of (1.2) – (1.4) follow from (1.1).

Ramanujan’s cubic continued fraction is defined by

$$v(q) := \frac{q^{1/3}}{1 + \frac{q + q^2}{1 + \frac{q^3 + q^6}{1 + \cdots}}}, \quad (1.6)$$

In his notebooks (Berndt, 1991; Ramanujan, 1957) and in his lost notebooks (Andrews & Berndt, 2005; Ramanujan, 1988), Ramanujan found that

$$v(q) = q^{1/3} \frac{\chi(-q)}{\chi^3(-q^3)} = q^{1/3} \frac{f(-q)f^3(-q^6)}{f(-q^2)f^3(-q^3)}. \quad (1.6)$$
Moreover, Ramanujan recorded several identities involving $v(q)$ (Berndt, 1991), namely,

\begin{align}
1 + \frac{1}{v(q)} & = \frac{\psi(q^{1/3})}{q^{1/3} \psi(q^3)},
1 - 2v(q) & = \frac{\varphi(-q^{1/3})}{\varphi(-q^3)},
1 + \frac{1}{v^3(q)} & = \frac{\psi^4(q)}{q^2 \psi^4(q^3)},
1 - 8v^3(q) & = \frac{\varphi^4(-q)}{\varphi^4(-q^3)},
\frac{1}{v(q)} + 4v^2(q) & = 3 + \frac{f^3(-q^{1/3})}{q^{1/3} f^3(-q^6)},
\end{align}

(1.7)

This continued fraction recently has been studied by several authors. Firstly, Chan (1995) derived the formula

\begin{equation}
v^3(q) = v(q^3) \frac{1 - v(q^3) + v^2(q^3)}{1 + 2v(q^3) + 4v^2(q^3)}.
\end{equation}

(1.8)

Next, Mahadeva Naika (2008) proved some identities

\begin{align}
\frac{1}{v^2(q)} - 2v(q) & = \left( 27 + \frac{f^{12}(-q^3)}{q^{1/3} f^{12}(-q^6)} \right)^{1/3},
\frac{1}{v^2(q)} - 2v(q) & = 3 + \frac{f^3(-q^{1/3})}{q^{2/3} f^3(-q^6)}.
\end{align}

(1.9)

Later, Chan (2010) proved that

\begin{align}
\frac{1}{v(q)} - 1 - 2v(q) & = \frac{f(-q^{1/3}) f(-q^{2/3})}{q^{1/3} f(-q^3) f(-q^6)},
\frac{1}{v^3(q)} - 7 - 8v^3(q) & = \frac{f^4(-q) f^4(-q^2)}{q f^4(-q^3) f^4(-q^6)}.
\end{align}

(1.10)
Hirschhorn and Roselin (2009) established the 2-, 3-, 4- and 6-dissections of Ramanujan’s cubic continued fraction and its reciprocal.

In this article, we will establish several new identities for \( v(q) \). In particular, Theorem 2.3 provides the results that can be used to give another proof of early work by Ramanujan in (1.7), Chan in (1.8), Mahadeva Naika in (1.9) and Chan in (1.10).

2. The Factorizations of the Ramanujan’s Cubic Continued Fraction

Firstly, we will state some identities for \( f(a, b) \) used in this literature.

**Lemma 2.1** (Berndt, 1991) Let \( U_n = a^{n(n+1)/2} b^{\frac{n(n-1)}{2}} \) and \( V_n = a^{n(n-1)/2} b^{\frac{n(n+1)}{2}} \) for each integer \( n \). Then

\[
f(U_1, V_1) = \sum_{r=0}^{k-1} U_r f\left( \frac{U_{k+r}}{U_r}, \frac{V_{k+r}}{U_r} \right),
\]

for every positive integer \( k \).

**Lemma 2.2** (Berndt, 1991) We have

\[
f(a, b) = f(b, a), \tag{2.2}
\]

\[
f(1, a) = 2 f(a, a^2). \tag{2.3}
\]

Using (1.1), the continued fraction \( v(q) \) can be re-expressed as

\[
v(q) = q^{1/3} \frac{f(q^3, q^3)}{f(q, q^2)}. \tag{2.4}
\]

Throughout this section, we let \( \zeta := e^{\pi i/3} \).

**Theorem 2.3** We have
\[
\frac{1}{\sqrt{v(q)}} + \sqrt{v(q)} = \frac{\psi\left(q^{1/2}\right)}{q^{1/6}} \sqrt{\frac{f(-q)}{f(-q^{2}) f(-q^{3}) f(-q^{6})}}, \tag{2.5}
\]

\[
\frac{1}{\sqrt{v(q)}} - \zeta \sqrt{v(q)} = \frac{\psi\left(-\zeta q^{1/3}\right)}{q^{1/6}} \sqrt{\frac{f(-q)}{f(-q^{2}) f(-q^{3}) f(-q^{6})}}, \tag{2.6}
\]

\[
\frac{1}{\sqrt{v(q)}} + \zeta^{2} \sqrt{v(q)} = \frac{\psi\left(q^{2} q^{1/3}\right)}{q^{1/6}} \sqrt{\frac{f(-q)}{f(-q^{2}) f(-q^{3}) f(-q^{6})}}, \tag{2.7}
\]

\[
\frac{1}{\sqrt{v(q)}} - 2\sqrt{v(q)} = \frac{\varphi(-q^{1/3})}{q^{1/6}} \sqrt{\frac{f(-q^{2})}{f(-q) f(-q^{3}) f(-q^{6})}}, \tag{2.8}
\]

\[
\frac{1}{\sqrt{v(q)}} + 2\zeta \sqrt{v(q)} = \frac{\varphi\left(\zeta q^{1/3}\right)}{q^{1/6}} \sqrt{\frac{f(-q^{2})}{f(-q) f(-q^{3}) f(-q^{6})}}, \tag{2.9}
\]

\[
\frac{1}{\sqrt{v(q)}} - 2\zeta^{2} \sqrt{v(q)} = \frac{\varphi(-q^{2} q^{1/3})}{q^{1/6}} \sqrt{\frac{f(-q^{2})}{f(-q) f(-q^{3}) f(-q^{6})}}. \tag{2.10}
\]

**Proof of (2.5).** By (2.4), the left hand side of (2.5) becomes

\[
\frac{1}{\sqrt{v(q)}} + \sqrt{v(q)} = \frac{f(q, q^{2}) + q^{1/3} f(q^{3}, q^{9})}{q^{1/6} \sqrt{f(q, q^{2}) f(q^{3}, q^{9})}} \tag{2.11}
\]

Using Jacobi’s triple product identity (1.1), we have

\[
f(q, q^{2}) f(q^{3}, q^{9}) = (-q; q^{3})_{\infty} (-q^{2}; q^{3})_{\infty} (q^{3}; q^{3})_{\infty} (-q^{3}; q^{3})_{\infty} (-q^{9}; q^{3})_{\infty} (q^{12}; q^{12})_{\infty} \tag{2.12}
\]

Putting \( k = 3, \ a = 1 \) and \( b = q^{1/3} \) in (2.1) together with (2.2) and (2.3), it follows that
\begin{align*}
f(1,q^{1/3}) &= f(q,q^2) + f(q^2,q) + q^{1/3} f(q^3,1), \\
2 f(q^{1/3},q) &= 2 f(q,q^2) + 2 q^{1/3} f(q^3,q^9), \\
\psi(q^{1/3}) &= f(q,q^2) + q^{1/3} f(q^3,q^9). \\
\end{align*}

(2.13)

Substituting (2.12) and (2.13) into (2.11), we obtain the result.

\textbf{Proof of (2.6).} Take \( k = 3 \), \( a = 1 \) and \( b = -\zeta q^{1/3} \) in (2.1).

\textbf{Proof of (2.7).} Put \( k = 3 \), \( a = 1 \) and \( b = \zeta^2 q^{1/3} \) in (2.1).

\textbf{Proof of (2.8).} By (2.4) and (2.12), we have

\[
\frac{1}{\sqrt[3]{v(q)}} - 2\sqrt[3]{v(q)} = \frac{(f(q,q^2) - 2q^{1/3} f(q^3,q^9))\sqrt{-q}}{q^{1/6}\sqrt{-q^{2}}f(-q^2)f(-q^3)f(-q^6)}. \\
\] 

(2.14)

Take \( k = 3 \), \( a = \zeta^2 \) and \( b = -\zeta q^{1/3} \) in (2.1) and obtain

\[
f(\zeta^2,-\zeta q^{1/3}) = f(q,q^2) + \zeta^2 f(q,q^2) - \zeta q^{1/3} f(q^3,1).
\]

Since \( \frac{\zeta}{1+\zeta^2} = 1 \), we arrive at

\[
\frac{f(\zeta^2,-\zeta q^{1/3})}{1+\zeta^2} = f(q,q^2) - 2q^{1/3} f(q^3,q^9). \\
\]

(2.15)

Using (1.1), we deduce that
\[
\frac{f(\zeta^2, -\zeta^{1/3})}{1 + \zeta^2} = \frac{(-\zeta^2; q^{1/3})_\infty \zeta^{1/3} (q^{1/3}; q^{1/3})_\infty (q^{1/3}; q^{1/3})_\infty}{1 + \zeta^2}
\]

\[
= f(-q^{1/3}) \prod_{k=1}^{\infty} \frac{1 + \zeta^2 q^{k/3}}{1 + q^{2k/3}}
\]

\[
= f(-q^{1/3}) \prod_{k=1}^{\infty} \frac{(1 - q^{2k})(1 - q^{k/3})}{(1 - q^{k})(1 - q^{2k/3})}
\]

\[
= \frac{f^2(-q^{1/3}) f(-q^2)}{f(-q^{2/3}) f(-q)}
\]

\[
= \frac{\varphi(-q^{1/3}) f(-q^2)}{f(-q)}.
\]

By (2.14), (2.15) and (2.16), we complete the proof of (2.8).

**Proof of (2.9).** Putting \( k = 3, \ a = -\zeta \) and \( b = q^{1/3} \) in (2.1) and using the fact that

\[
f(-\zeta, q^{1/3}) = \frac{(1 - \zeta) \varphi(\zeta q^{1/3}) f(-q^2)}{f(-q)},
\]

we finish the proof of (2.9).

**Proof of (2.10).** Taking \( k = 3, \ a = \zeta^2 \) and \( b = q^{1/3} \) in (2.1) and employing

\[
f(\zeta^2, q^{1/3}) = \frac{(1 + \zeta^2) \varphi(-\zeta^2 q^{1/3}) f(-q^2)}{f(-q)}
\]

the proof of (2.10) is complete.

**Theorem 2.4** We have

\[
\frac{1}{v(q)} - 1 - 2v(q) = \frac{f(-q^{1/3}) f(-q^{2/3})}{q^{1/3} f(-q^2) f(-q)}, \quad (2.17)
\]
Proof of (2.17). Multiply (2.5) and (2.8), we complete the proof of (2.17).

Proof of (2.18). We observe that

\[
\frac{1}{v(q)} - 1 + v(q) = \left( \frac{1}{\sqrt{v(q)} - \zeta \sqrt{v(q)}} - \frac{1}{\sqrt{v(q)} + \zeta \sqrt{v(q)}} \right). 
\]

Employing (2.6) and (2.7) together with (1.5), we arrive at

\[
\frac{1}{v(q)} - 1 + v(q) = \frac{\psi\left( -\zeta q^{1/3} \right) \psi\left( \zeta^2 q^{1/3} \right) \chi(-q)}{q^{1/3} f(-q^3) f(-q^6)}. 
\]

We find that

\[
\psi\left( -\zeta q^{1/3} \right) \psi\left( \zeta^2 q^{1/3} \right) = \prod_{k=0}^{\infty} \frac{\left( 1 - \zeta q^{2k+2} q^{(2k+2)/3} \right) \left( 1 - \zeta^2 q^{2k+1} q^{2k+2} q^{(2k+2)/3} \right)}{\left( 1 - \zeta^2 q^{2k+1} q^{2k+2} q^{(2k+2)/3} \right) \left( 1 - \zeta^{2k+2} q^{(2k+2)/3} \right) \left( 1 - \zeta^{4k+2} q^{(2k+2)/3} \right) \left( 1 - \zeta^{4k+1} q^{(2k+2)/3} \right)} 
\]

\[
= \prod_{k=0}^{\infty} \frac{1 - \zeta^{4k+1} q^{(2k+1)/3} + \zeta^{2(2k+1)} q^{2(2k+2)/3} + q^{4k+1/3}}{1 - \zeta^{4k+1} q^{(2k+1)/3} + \zeta^{2(2k+1)} q^{2(2k+2)/3} + q^{4k+1/3}}. 
\]

Since

\[
\zeta^{4k+1} + \zeta^{2(2k+1)} = \begin{cases} 
2 & \text{if } 3 \mid k + 1, \\
-1 & \text{otherwise}, 
\end{cases}
\]

and
the equation (2.21) becomes

\[
\psi\left(-\zeta q^{1/3}\right)\psi\left(\zeta^2 q^{1/3}\right) = \prod_{k=0}^{\infty} \frac{1-2q^{2^{k+2}} + q^{4^{k+4}}}{1-2q^{2^{k+1}} + q^{4^{k+2}}} \left(1-q^{6^{k+2}/3}\right) \left(1-q^{12^{k+8}/3}\right) \left(1-q^{(6k+4)/3} + q^{(12k+8)/3}\right) \left(1+q^{(6k+4)/3} + q^{(12k+8)/3}\right) 
\]

By (1.4), it follows that

\[
\psi\left(-\zeta q^{1/3}\right)\psi\left(\zeta^2 q^{1/3}\right) = \psi\left(q^{1/3}\right)f\left(-q^2\right) \cdot \psi\left(q^{1/3}\right)f\left(-q^2\right). 
\] (2.22)

Substituting (2.22) into (2.20), the proof is complete.

**Proof of (2.19).** We find that

\[
\frac{1}{v(q)} + 2 + 4v(q) = \left(\frac{1}{\sqrt{v(q)}} + 2\zeta \sqrt{v(q)}\right)\left(\frac{1}{\sqrt{v(q)}} - 2\zeta^2 \sqrt{v(q)}\right).
\]

Utilizing (2.9) and (2.10), we deduce that

\[
\frac{1}{v(q)} + 2 + 4v(q) = \frac{\phi(\zeta^{1/3})\phi(-\zeta^2 q^{1/3})f\left(-q^2\right)}{q^{1/3}f\left(-q\right)f\left(-q^3\right)f\left(-q^5\right)}. 
\] (2.23)

For the numerator of (2.23), we see that...
\[
\phi(z q^{1/3}) \phi(-z^{-1} q^{1/3}) = \frac{(-z q^{1/3}; -z q^{1/3})_{\infty}}{(z q^{1/3}; z q^{1/3})_{\infty}} \frac{(z^2 q^{1/3}; z^2 q^{1/3})_{\infty}}{(-z^2 q^{1/3}; -z^2 q^{1/3})_{\infty}} \\
= \prod_{k=1}^{\infty} \left( 1 - \left( \frac{z q^{1/3}}{1} \right)^{k} \right) \left( 1 - \left( \frac{z^{-1} q^{-1/3}}{1} \right)^{k} \right) \\
= \prod_{k=1}^{\infty} \frac{1 - \left( \frac{z^2 q^{1/3}}{1} \right)^{k} \left( \frac{-z^{-2} q^{-1/3}}{1} \right)^{k}}{1 + \left( \frac{z^{2} q^{1/3}}{1} \right)^{k} + \left( \frac{-z^{-2} q^{-1/3}}{1} \right)^{k}} \\
= \prod_{k=1}^{\infty} \frac{1 - \left( z^{2k} + (-z)^{k} \right) q^{k/3} + q^{2k/3}}{1 + \left( z^{2k} + (-z)^{k} \right) q^{k/3} + q^{2k/3}}.
\]

Since \( z^{2k} + (-z)^{k} = 2 \) if \( 3 | k \) and \( z^{2k} + (-z)^{k} = -1 \) if \( 3 \nmid k \), it follows that

\[
\phi(z q^{1/3}) \phi(-z^{-1} q^{1/3}) = \prod_{k=0}^{\infty} \left( 1 - \frac{q^{1/3 + k}}{1 + q^{1/3 + k}} \right) \left( 1 - \frac{q^{2(1/3 + k)}}{1 + q^{2(1/3 + k)}} \right) \\
= \prod_{k=0}^{\infty} \left( 1 - \frac{q^{1/3 + k}}{1 - q^{1/3 + k}} \right) \left( 1 - \frac{q^{3/2 + k}}{1 + q^{3/2 + k}} \right) \\
= \frac{(q;q)^{2}}{(1-q)_{\infty}} \frac{(q^3;q^3)_{\infty}(q^2;q^3)_{\infty}}{(-q;q)_{\infty}(-q^2;q)_{\infty}} \frac{(-q^{1/3};q)_{\infty}(-q^{2/3};q)_{\infty}}{(-q^{1/3};q)_{\infty}(-q^{2/3};q)_{\infty}} \\
= \frac{(q;q)^{8}}{(q^6;q^6)_{\infty}} \frac{(-q^2;q)_{\infty}(q^2;q)_{\infty}}{(-q^2)_{\infty}(q^2)_{\infty}} \frac{(-q^{2/3};q^{2/3})_{\infty}}{(-q^{2/3};q^{2/3})_{\infty}} \\
= \frac{f^8(-q)f(-q^6)f(-q^{2/3})}{f^4(-q^2)f(-q^2)f(-q^{1/3})}.
\]

(2.24)

Substituting (2.24) into (2.23), we complete the proof.

**Corollary 2.5** We have

\[
\frac{1}{\sqrt{v^3(q)}} + \sqrt{v^3(q)} = \frac{\psi^4(q)}{q^{1/2} \psi(q^3)} \sqrt{\frac{z^3(-q)}{f^3(-q^3)f^3(-q^6)}},
\]

(2.25)
\[
\frac{1}{\sqrt{v^3(q)}} - 8\sqrt{v^3(q)} = \frac{f^4(-q)}{qf^4(-q^2)} \sqrt{\frac{\chi^6(-q)}{f^4(-q^3) f^3(-q^6)}}, \tag{2.26}
\]

\[
\frac{1}{v^3(q)} - 7 - 8v^3(q) = \frac{f^4(-q)f^4(-q^3)}{qf^4(-q^3)f^4(-q^6)}, \tag{2.27}
\]

\[
v^3(q) = v(q^3)\frac{1 - v(q^3) + v^2(q^3)}{1 + 2v(q^3) + 4v^2(q^3)}. \tag{2.28}
\]

**Proof of (2.25).** The identity follows from the factorization

\[
\frac{1}{\sqrt{v^3(q)}} + \sqrt{v^3(q)} = \left(\frac{1}{\sqrt{v(q)}} + \sqrt{v(q)}\right)\left(\frac{1}{v(q)} - 1 + v(q)\right).
\]

**Proof of (2.26).** The identity can be obtained from

\[
\frac{1}{\sqrt{v^3(q)}} - 8\sqrt{v^3(q)} = \left(\frac{1}{\sqrt{v(q)}} - 2\sqrt{v(q)}\right)\left(\frac{1}{v(q)} + 2 + 4v(q)\right).
\]

**Proof of (2.27).** Note that

\[
\frac{1}{v^3(q)} - 7 - 8v^3(q) = \left(\frac{1}{\sqrt{v^3(q)}} + \sqrt{v^3(q)}\right)\left(\frac{1}{\sqrt{v^3(q)}} - 8\sqrt{v^3(q)}\right).
\]

Multiply (2.25) and (2.26) and then obtain

\[
\frac{1}{v^3(q)} - 7 - 8v^3(q) = \frac{f^4(-q)\psi^4(q)\chi^4(-q)}{q\psi(q^3)\chi(-q^3)f^4(-q^3)f^3(-q^6)}.
\]

Since \(\psi^4(q)\chi^4(-q) = f^4(-q^2)\) and \(\psi(q^3)\chi(-q^3) = f(-q^6)\), the result follows immediately.

**Proof of (2.28).** Using (2.18), (2.19) and (1.6), we arrive at
\[ v(q^3) \frac{1 - v(q^3) + v^2(q^3)}{1 + 2v(q^3) + 4v^2(q^3)} = q \left( \frac{\varphi(-q)}{\psi(q)} \right) \left( \frac{\psi^4(q^3) \chi(-q^3)}{\varphi^3(-q^3) f(-q^6)} \right). \] (2.29)

Utilizing (1.3), (1.4) and (1.5), we get

\[ \frac{\varphi(-q)}{\psi(q)} = \frac{f^3(-q)}{f^3(-q^2)}, \] (2.30)

and

\[ \frac{\psi^4(q^3) \chi(-q^3)}{\varphi^3(-q^3) f(-q^6)} = \frac{f^3(-q^6)}{f^3(-q^5)}. \] (2.31)

Hence substituting (2.30) and (2.31) into (2.29) together with (1.6), we eventually conclude that

\[ v(q^3) \frac{1 - v(q^3) + v^2(q^3)}{1 + 2v(q^3) + 4v^2(q^3)} = q \left( \frac{f^3(-q)}{f^3(-q^2)} \right) \frac{f^3(-q^6)}{f^3(-q^5) f(-q^6)} = v^3(q). \]

The following corollary mainly contains Ramanujan’s results of \( v(q) \).

**Corollary 2.6** We have
\[ 1 + v(q) = \frac{\psi(q^{1/3})\chi(-q)}{\phi(-q^{3})}, \]
\[ \frac{1}{v(q)} + 1 = \frac{\psi(q^{1/3})}{q^{1/3}\psi(q^{3})}, \]
\[ 1 - 2v(q) = \frac{\phi(-q^{1/3})}{\phi(-q^{3})}, \]
\[ \frac{1}{v(q)} - 2 = \frac{\phi(-q^{1/3})}{q^{1/3}\chi(-q)\psi(q^{3})}, \]
\[ 1 + v^3(q) = \frac{\phi(-q)\psi^3(q)}{\phi^3(-q^{3})\psi(q^{3})}, \]
\[ \frac{1}{v^3(q)} + 1 = \frac{\psi^4(q)}{q\psi^3(q^{3})}, \]
\[ 1 - 8v^3(q) = \frac{\phi^3(-q)}{\phi^3(-q^{3})}, \]
\[ \frac{1}{v^3(q)} - 8 = \frac{\phi^3(-q)\psi(q)}{q\phi(q^{3})\psi^3(q^{3})}. \]

**Proof.** These results are immediate consequences of Theorem 2.3 and Corollary 2.5.

**Corollary 2.7** We have

\[ \frac{1}{v(q)} + 4v^2(q) = \left(27 + \frac{f^{12}(-q)}{qf^{12}(-q^{3})}\right)^{1/3}, \] (2.32)

\[ \frac{1}{v(q)} + 4v^2(q) = 3 + \frac{f^3(-q^{1/3})}{q^{1/3}f^3(-q^{3})}, \] (2.33)

\[ \frac{1}{v^2(q)} - 2v(q) = \left(27 + \frac{f^{12}(-q^2)}{q^2f^{12}(-q^{6})}\right)^{1/3}, \] (2.34)

\[ \frac{1}{v^2(q)} - 2v(q) = 3 + \frac{f^3(-q^{2/3})}{q^{2/3}f^3(-q^{6})}. \] (2.35)

**Proof.** Employing Theorem 2.3, Corollary 2.5 and Corollary 2.6, the identities (2.32) - (2.35) follow immediately from the following factorizations.
\[
\left( \frac{1}{v(q)} + 4v^2(q) \right)^3 - 27 = (1 + v(q)) \left( \frac{1}{\sqrt{v^3(q)}} - 8\sqrt{v^3(q)} \right)^2,
\]
\[
\frac{1}{v(q)} + 4v^2(q) - 3 = (1 + v(q)) \left( \frac{1}{\sqrt{v(q)}} - 2\sqrt{v(q)} \right)^2,
\]
\[
\left( \frac{1}{v^2(q) - 2v(q)} \right)^3 - 27 = \frac{1}{\sqrt{v^3(q)}} \left( \frac{1}{\sqrt{v^3(q)}} + \sqrt{v^3(q)} \right)^2 \left( \frac{1}{\sqrt{v^3(q)}} - 8\sqrt{v^3(q)} \right),
\]
\[
\frac{1}{v^2(q) - 2v(q)} - 3 = \frac{1}{\sqrt{v(q)}} (1 + v(q))^2 \left( \frac{1}{\sqrt{v(q)}} - 2\sqrt{v(q)} \right),
\]
respectively.

References


